Application of character sums to mathematical cryptography

-- Some examples in multivariate quadratic cryptography --

Tomohiro Harayama

Security Fundamentals Group
Information Security Research Center
National Institute of Information
and Communications Technology, JAPAN

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Contents

- MQ problems (NP-complete) and Cryptosystems
- Weil sums and Weil sum algorithm
- Number of solutions of equations (Formula? or #P-complete)
- Weak Dembowski-Ostrom Polynomials (Potential existential forgery if used in Digital Signature)
- Summary (How can we do design-analysis-redesign ?)

Multivariate Quadratic (MQ) Problems and Cryptosystems

Oneway function

X, Z :sets. $f : X \to Z$ is called a ***oneway function*** if:

- It is "easy" to compute $z = f(x)$ for $\forall x \in X$

- For essentially all elements $z \in Im(f)$ the image by it is computationally infeasible to find any $x \in X$ such that $z = f(x)$

It is well known that:

Existence of $f : X \to Z \Rightarrow$ $\mathrm{P} \neq \mathcal{NP}$.

Cryptographic Reference Problems

- Integer factorization (e.g., RSA problem)
- Discrete logarithm (e.g., $\mathbb{Z}_p$, $\mathcal{EC}(\mathbb{F}_q)$)
- Lattice reduction (e.g., SVP problem)
- *Systems of multivariate quadratic polynomials over finite fields*
- Others

MQ (NP-complete problem)

$(y_1, \ldots, y_n) \in \mathbb{F}_2^n$ ← **Easy** / **Very hard** → $(x_1, \ldots, x_n) \in \mathbb{F}_2^n$

Hashed message — **Signature**

$$
\begin{cases}
y_1 = \sum_{1 \le i \le j \le n} \alpha_{i,j}^{(1)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(1)} x_i + \gamma_i^{(1)} \\
y_2 = \sum_{1 \le i \le j \le n} \alpha_{i,j}^{(2)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(2)} x_i + \gamma_i^{(2)} \\
\vdots \\
\vdots \\
y_n = \sum_{1 \le i \le j \le n} \alpha_{i,j}^{(n)} x_i x_j + \sum_{i=1}^{n} \beta_i^{(n)} x_i + \gamma_i^{(n)}
\end{cases}
$$

Birthday Attack on Hash Function

Let be $h : D \rightarrow R$ a hash function from domain D to R. Then, birthday attack is to find at least one collision $x, x' \in D$ s.t. $x \neq x'$ and $h(x) = h(x')$.

Note: a sorted list H
For i = 0 to |R|:
*** Select x uniformly at random with replacement from D.**
*** Compute y = h(x) and store pair (y, x) in H (sorted in y).**
*** If (some pair (y', x') in H) s.t.**
y' = y and x' != x,
then return (x, x', y), i // collision !
Return Null

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Expected number of steps (memory and time) is $O(\sqrt{|R|})$ because the number of ordered pairs (i_1, i_2) with $1 \leq i_1 \leq i_2 \leq i$ grows **quadratically** w.r.t. i (i.e., $\frac{i(i-1)}{2}$), so does probability for having collisions.

Birthday Attack on Digital Signature based on $MQ(p,n,n)$-Trapdoor $\mathbb{F}_p^n \to \mathbb{F}_p^n$

A ***Birthday Attack*** is to find at least one pair

$sig, sig' \in D$ s.t.

$$sig \neq sig' \text{ and}$$

$$F(sig) = F(sig') = message.$$

Note: a sorted list H.
For k = 0 to p^n:
 * Select sig uniformly at random
   with replacement from GF(p^n).
 * Compute msg = F(sig)
   store pair (msg, sig) in H.
 * If some (msg', sig') s.t.
   msg = msg' and sig' != sig return (sig, sig', msg) k;
Return Null.

Is it really $O(\sqrt{2^n})$ for all $F \in \mathcal{MQ}(2^n, n, n)$?

Kipnis-Shamir's Lemma

Let *standard linear bijection* $\phi : \mathbb{F}_{q^n} \to \mathbb{F}_q^n$.

For $\forall$ multi-polys $(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n))$ over $\mathbb{F}_q$,

$\exists$ uni-poly $f \in \mathbb{F}_{q^n}[x]$:

$$f(x) = \sum_{i=1}^{D} a_i x^{q^{\alpha_i}+q^{\beta_i}} + \sum_{j=1}^{L} b_j x^{q^{\gamma_j}} + c,$$

such that: $\forall (x_1, \ldots, x_n) \in F_q^n$

$$\phi \circ f \circ \phi^{-1}(v_1, \ldots, v_n) = (P_1(v_1, \ldots, v_n), \ldots, P_n(v_1, \ldots, v_n))$$

Central Polynomial

Given $(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n))$ of $MQ(q, n, n)$. A ***central polynomial*** over $\mathbb{F}_q$: is of form:

$$f(x) = \sum_{i=1}^{D} a_i x^{q^{\alpha_i}+q^{\beta_i}} + \sum_{j=1}^{L} b_j x^{q^{\gamma_j}} + c.$$

$$F : \mathbb{F}_q^n \to \mathbb{F}_q^n \overset{Kipnis-Shamir}{\Leftrightarrow} f : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n} \text{ (central)}$$

DO Polynomial

Homogeneous $(P_1(x_1, \ldots, x_n), \ldots, P_n(x_1, \ldots, x_n))$ of $MQ(q, n, n)$. A ***DO polynomial*** over $\mathbb{F}_q$: is of form:

$$f(x) = \sum_{i=1}^{D} a_i x^{q^{\alpha_i}+q^{\beta_i}}$$

$F : \mathbb{F}_q^n \to \mathbb{F}_q^n \overset{Kipnis-Shamir}{\Leftrightarrow} f : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ DO **Homogeneous**

Weil Sum and Weil Sum Algorithm

Simplified Central Polynomial

$f(x)$: ***central polynomial*** over $\mathbb{F}_q$:
$(f(0) = 0,\ 1 \leq i \leq D)$ s.t.

$$t_i, y_i, s_i \in \mathbb{Z},$$
$$A_i \text{ s.t. } A_i^{p^{t_i}} = a_i \in \mathbb{F}_q,$$
$$b = \sum_{j=1}^{L} b_j^{p^{e-\gamma_j}}$$
$$\beta_i \in \mathbb{F}_q \text{ s.t. } t_i \equiv \beta_i - \beta_1 \mod n,$$
$$y_i = n - s_i \ (2 \leq i \leq D),$$
$$s_i = \alpha_i - \beta_i \geq 0.$$

Simplified Central Polynomial

Weil sum values are **equivalent** for general and simplified central polynomials

$$f(x) = \sum_{i=1}^{D} A_i x^{p^{s_i}+1} + b^{p^{\beta_1}} x$$

so that we work on the Weil sum:

$$S(a_1, \ldots, a_D, b_1, \ldots, b_L)$$
$$= \sum_{x \in \mathbb{F}_q} \chi_1\left(\sum_{i=1}^{D} A_i x^{p^{s_i}+1} + b^{p^{\beta_1}} x\right)$$

Auxiliary Linearized Polynomial

The computational complexity of Weil sum algorithm is governed by the **dimension**.

The product of Weil sum $|S|$ is

$$|S|^2 = q \sum_{T_D(w)=0, w\in\mathbb{F}_q} \chi_1\Big(\sum_{i=1}^{D} A_i w^{p^{s_i}+1} + b^{p^{\beta_1}} w\Big).$$

The index w of the outer sum runs throughout the **set of roots** in $\mathbb{F}_q$ of a linearized polynomial:

$$T_D(w) = A_1^{p^{s1}} w^{p^{2s_1}} + A_1 w + \sum_{i=2}^{D} \big[A_i^{p^{s1}} w^{p^{s_1+s_i}} + (A_i w)^{p^{s_1+y_i}}\big].$$

Root of Auxiliary Linearized Polynomial

$$T_D(w) = A_1^{p^{s1}} w^{p^{2s_1}} + A_1 w + \sum_{i=2}^{D} [A_i^{p^{s1}} w^{p^{s_1+s_i}} + (A_i w)^{p^{s_1+y_i}}].$$

$$\varepsilon = \gcd_{2 \le i \le D}(2s_1, s_1 + s_i, s_1 + y_i, n).$$

The **set of roots** is:

- a linear subspace $\subset \mathbb{F}_q$ over $\mathbb{F}_{p^\varepsilon}$
- $\cong \mathbb{F}_{p^{t\varepsilon}}$ for some integer $t \in \mathbb{Z}$.

The dimension $t\epsilon$ of the roots of $T_D(w)$ is defined by portions of exponents $2s_1, s_1 + s_i, s_1 + y_i$ and n.

Character ($p = 2$)

- Let $\mathbb{F}_q$ be of characteristic $p = 2$. Then, for any $u \in \mathbb{F}_q$, $\chi_1(u) = \exp(2\pi i Tr(u)/2)$ is real.

- Let $p = 2$ and

$$S = \sum_{x \in \mathbb{F}_q} \chi_1\left(\sum_{i=1}^{D} A_i x^{p^{s_i}+1} + b^{p^{\beta_1}} x\right)$$:

the Weil sum of central polynomial. Then, S is real and $|S|^2 = S^2$.

Summing 1 or -1 for at most q times.

Weil Sum Algorithm ($p = 2$)

INPUT $f(x) = \sum_{i=1}^{D} a_i x^{2^{\alpha_i}+2^{\beta_i}} + \sum_{i}^{L} b_i x^{2^{\gamma_i}}$: central polynomial in $\mathbb{F}_{2^n}[x]$.
OUTPUT $|S|$: the absolute value of Weil sum S of $f(x)$.

---

1. Compute $T_D(x) \in \mathbb{F}_{2^n}[x]$ (Suppose the rank of the kernel is l).
2. Compute the basis $\{\gamma_1, \ldots, \gamma_l\}$ of $ker(T_D)$.
3. Let U be $0 \in Z$.
4. Compute $\gamma_{i,j_1,j_2} = Tr(A_i \gamma_{j_1} \gamma_{j_2}^{2^{s_i}})$ for $1 \leq i \leq D, 1 \leq j_1, j_2 \leq l$
5. Compute $\rho_j = Tr(b^{2^{\beta_1}} \gamma_j)$ for $1 \leq j \leq l$.
6. For each $(x_1, \ldots, x_l) \in \mathbb{F}_2^l$, evaluate:

$$C_{(x_1,\ldots,x_l)} = \sum_{i=1}^{D} \sum_{j_1=1}^{l} \sum_{j_2=1}^{l} x_{j_1} x_{j_2} \gamma_{i,j_1,j_2} + \sum_{j=1}^{l} x_j \rho_j \in \mathbb{F}_2$$

   and set $U = U + C_{(x_1,\ldots,x_l)}$. (Note: **integer addition**.)
7. Return $2^{n/2}\sqrt{2^l - 2U}$. (Note: **absolute value**)

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Go through each element in kernel

$$C_{(x_1,\ldots,x_l)} = \sum_{i=1}^{D} \sum_{j_1=1}^{l} \sum_{j_2=1}^{l} x_{j_1} x_{j_2} \gamma_{i,j_1,j_2} + \sum_{j=1}^{l} x_j \rho_j \in \mathbb{F}_2$$

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Time complexity: $O(C_D L l^2 (n^3 + 2^l))$

Number of Solutions of Equations

Batched Solutions

Set $\delta = \gcd(s_1, \ldots, s_D, n)$. Let $f(x)$ be a simplified ***Dembowski-Ostrom polynomial*** $\sum_{i=1}^{D} A_i x^{p^{s_i}+1}$ over with $\mathbb{F}_q$ each s_i/δ odd and $g(x,y) = f(x) - y^{p^\delta} + y$ the bivariate polynomial. Then, the number of solutions $N(g(x,y))$ of the bivariate equation $g(x,y) = 0$ is estimated as:

$$N \equiv -1 \mod p^\delta + 1.$$

Condition s_i/δ: odd

$\Longrightarrow$ congruential estimate of $N(f(x) - y^{p^\delta} + y)$

GCD and Exponents

$n, s_i\ (1 \le i \le D)$: non-negative integers with

$$\delta = \gcd(s_1, \ldots, s_D, n).$$

s_i/δ : odd for each $1 \le i \le D$ forces s_i to be positive.

Assume that n/δ is even. Then we have:

$$\left(\frac{2^{s_i}+1}{2^{\delta}+1}, 2^{\delta} - 1\right) = 1.$$

Proof (idea):

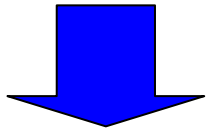
$$2^{s_i} + 1 = (2^{\delta} + 1)(2^{(s_i/\delta - 1)\delta} - 2^{(s_i/\delta - 2)\delta} + \ldots - 2^{\delta} + 1).$$

even odd …….. odd, even

Bivariate Equation. Emulation Condition

Let $f(x)$ a DO poly $\sum_{i=1}^{D} A_i x^{p^{s_i}+1} over \mathrm{F}_q$. $\delta = (s_1, \ldots, s_D)$.

n/δ is even,
$\delta = (s_i, e)$ for each i,
s_i/δ is odd for each i, and
2δ divides $s_i - s_j$ for all $i \neq j$.



$$N(f(x,y)) = q + (p^{\delta} - 1)S.$$

Relating # solutions N to Weil sum

$$S = \sum_{x \in \mathbb{F}_q} \chi_1\left(\sum_{i=1}^{D} A_i x^{p^{s_i}+1} + b^{p^{\beta_1}} x\right)$$

Simplified Weil Sum Algorithm (p=2)

INPUT $f(x) = \sum_{i=1}^{D} A_i x^{2^{s_i}+1}$: DO poly with EC.
OUTPUT S: Weil sum of $f(x)$

---

1. Compute $T_D(x) \in \mathbb{F}_{2^n}[x]$ (Suppose the rank of the kernel is l).
2. Compute the basis $\{\gamma_1, \ldots, \gamma_l\}$ of $ker(T_D)$.
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   and set $U = U + C_{(x_1,\ldots,x_l)}$. (Note: **integer addition**.)

6. If $|S| = 2^{n/2}\sqrt{2^l - 2U}$ satisfies $2(1 - |S|\ |) \equiv 0 \mod 2^{\delta} + 1$ return $|S|$. Otherwise return $-|S|$.

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Go through each element in kernel

6. If $|S| = 2^{n/2}\sqrt{2^l - 2U}$ satisfies $2(1 - |S|\,|) \equiv 0 \mod 2^{\delta} + 1$ return $|S|$. Otherwise return $-|S|$.

Time complexity: $O(D\, l^2 (n^3 + 2^l))$

Large # of Solutions: Potential cryptanalytic implications

$$S = \{(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n} \mid \sum_{i=1}^{D} A_i x^{2^{s_i}+1} = y^{2^\delta} - y\}$$

If **S is large**, we pick up random $(x, y) \in \mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ and the probability that

$x \to \sum_{i=1}^{D} A_i x^{2^{s_i}+1}$ **=?** $y^{2^\delta} - y \leftarrow y$

is higher.

Oneway: Hard to invert.

Meet-in-the-Middle?

Linear: easy to Invert ?

Weak Dembowski-Ostrom Polynomials

Existential Forgery (if Signature Scheme)

An adversary is able to forge a signature of **least one** message over which the adversary has little or no control.

Suppose we consider a key subspace of:

$$\mathcal{K} = \{ \sum_{i=1}^{D} A_i x^{2^{s_i}+1} \}$$

Ideally, **none** of key in $\mathcal{K}$ should be weak.

Linearized Binomial Attack

Let $f(x)$ be a secret Dembowski-Ostrom polynomial:

$$f(x) = \sum_{i=1}^{D} A_i x^{p^{s_i}+1} = A_1 x^{p^{s_1}+1} + \cdots + A_D x^{p^{s_D}+1},$$

where $D \geq 1$, $A_i \in \mathbb{F}_q$ for $1 \leq i \leq D$ and

$$0 \leq s_1 < s_2 < \ldots < s_D \leq q-1.$$

With $D \geq 1$ and $\delta = (s_1, \ldots, s_D, n)$ such that:

$$\mathcal{K} = \{\sum_{i=1}^{D} A_i x^{2^{s_i}+1}, n/\delta \text{ even}, \delta = (s_i, n), s_i/\delta \text{ odd}, 2\delta \text{ divides } s_i - s_j\}.$$

$\mathcal{K}$ is the key space of our attack.

A *linearized binomial attacker* against $MQ(2,n,n)$-trapdoor

$$F(x) = (P_1(x_1,\ldots,x_n),\ldots,P_n(x_1,\ldots,x_n)).$$

Randomly guess the value of the unknown $\delta = (s_1,\ldots,s_D,n)$ from $\{1,\ldots,n\}$. This δ allows the adversary to fix a linearized binomial $L(y) = y^2 - y$ in $\mathbb{F}_q[y]$.

We denote by $Im(L)$ the image of the mapping L over $\mathbb{F}_q$.

1. Generate $\frac{T}{t}2^{\frac{n-\delta}{2}}$ random elements $x \in \mathbb{F}_q$ and obtain the list:

$$\{f(x_1), f(x_2), \ldots, f(x_{\frac{T}{t}2^{\frac{n-\delta}{2}}})\}.$$

2. Generate $2^{\frac{n-\delta}{2}}$ random elements $z \in Im(L)$ to obtain the list:

$$\{z_1,\ldots,z_{2^{\frac{n-\delta}{2}}}\}.$$

3. Search for a coincidence $f(x_j) = z_i$ for some i,j in the two lists.

Complexity: linearized binomial attacker must generate

$$\frac{T}{t}2^{\frac{n-\delta}{2}} > 2^{\frac{n-\delta}{2}}$$

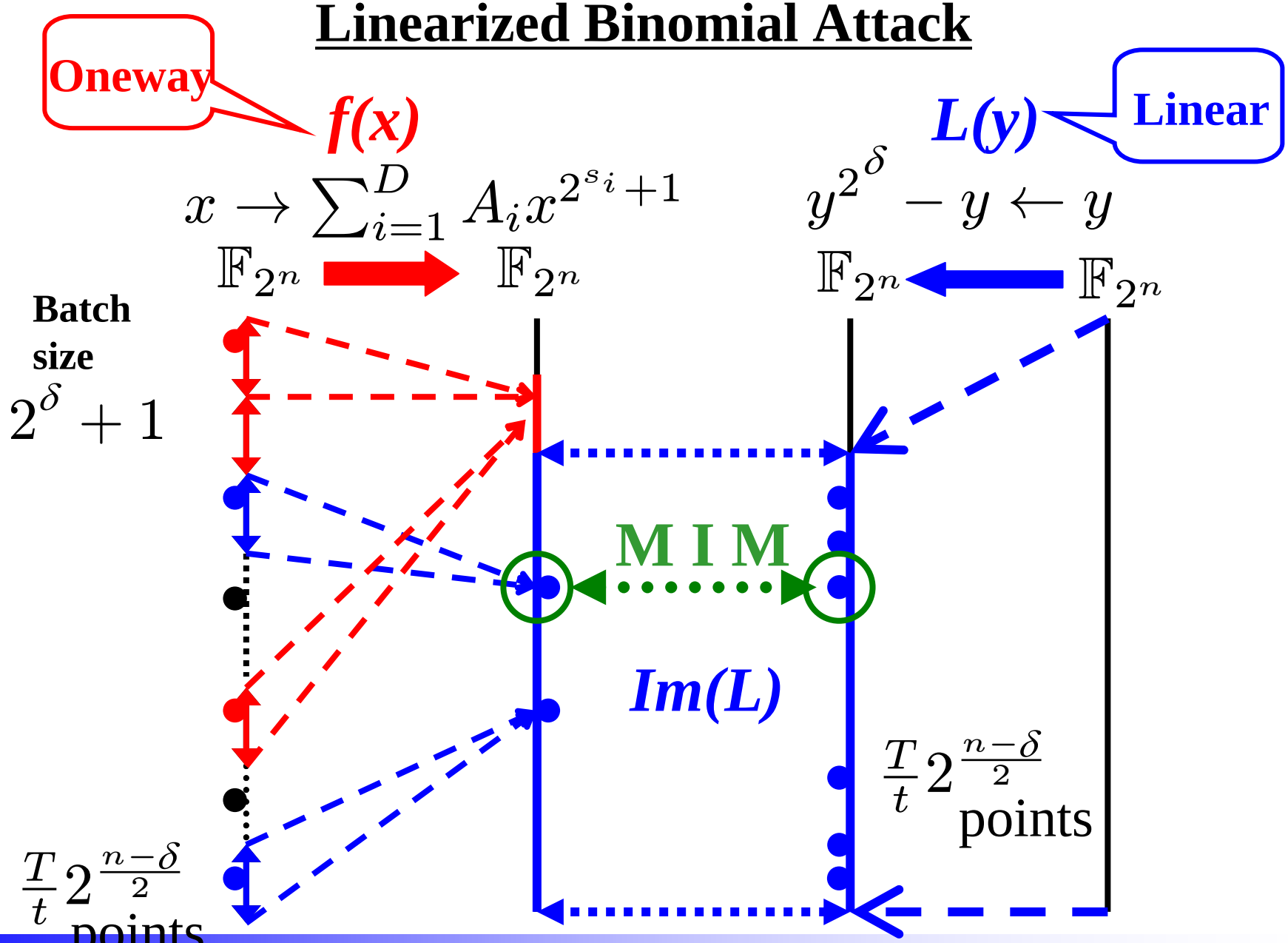
elements x's for their images $f(x)$'s in order to obtain the list

$$\{f(x_1), f(x_2), \ldots, f(x_{\frac{T}{t}2^{\frac{n-\delta}{2}}})\}$$

in which at least $2^{n/\delta}$ elements are expected to be inside $Im(L)$.
δ is guessed from $\{1, \ldots, n\}$.

The total time complexity: $O(n \times \frac{T}{t}2^{\frac{n-\delta}{2}})$.

Linearized Binomial Attack



n/4

$$f(x) = A_1 x^{2^{s_1}+1} + A_2 x^{2^{s_2}+1} \text{ (i.e. } D = 2)$$

For every integer $i \geq 1$, define the even extension degree n of $\mathbb{F}_{2^n}$ over $\mathbb{F}_2$ and (s_1, s_2): exponents of $f(x)$ such as:

$$\begin{cases} n = 4i, \\ s_1 = i, \\ s_2 = 3i. \end{cases}$$

where $\begin{cases} \delta = (s_1, n) = (s_2, n) = i, \\ n/\delta = 4 : \text{ even }, \\ s_1/\delta = 1 \text{ odd }, s_2/\delta = 3 \text{ odd and} \\ 2\delta = 2i \text{ divides } |s_2 - s_1|. \end{cases}$

Thus, we have $\delta = n/4$.

Weak DO Polynomials (*D=2, delta = n/4*)

$f(x) = x^{2^{s1}+1} + x^{2^{s2}+1}$: DO-poly in $F(2^n)[x]$ (n from 8 to 24)

$n = 8, (s_1, s_2) = (2, 6)$:

$$f(x) = x^{2^2+1} + x^{2^6+1} = x^5 + x^{65} \text{ in } \mathbb{F}_{2^8}[x].$$

$n = 12, (s_1, s_2) = (3, 9)$:

$$f(x) = x^{2^3+1} + x^{2^9+1} = x^9 + x^{513} \text{ in } \mathbb{F}_{2^8}[x].$$

$n = 16, (s_1, s_2) = (4, 12)$:

$$f(x) = x^{2^4+1} + x^{2^{12}+1} = x^{17} + x^{4097} \text{ in } \mathbb{F}_{2^{16}}[x].$$

$n = 20, (s_1, s_2) = (5, 15)$:

$$f(x) = x^{2^5+1} + x^{2^{15}+1} = x^{33} + x^{32769} \text{ in } \mathbb{F}_{2^{20}}[x].$$

$n = 24, (s_1, s_2) = (6, 18)$:

$$f(x) = x^{2^6+1} + x^{2^{18}+1} = x^{65} + x^{262145} \text{ in } \mathbb{F}_{2^{24}}[x].$$

Summary (How can we do design-analysis-redesign ?)

1. Design: Given a cryptosystem as it is.
2. Analysis
    1. Any formula ? (e.g., # of solutions of polynomial equations)
    2. Any algorithm ? (e.g., Weil sum evaluation algorithm)
    3. Any relations ? (e.g., Weil sum value to the number of solutions)
    4. Any attack? (e.g., Attack algorithm to characterize “weak” keys)
    5. Any experiment ? (e.g., Proof of existence of such weak keys. If exists on smaller parameters (e.g. n), try to derive some formula for n of any size)
3. (Try to) Redesign it (e.g., Think how to eliminate weak key class from trapdoor structure or key generation algorithm).

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