Topics on odd perfect numbers with a special structure

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Introduction

As usual, we denote by $\sigma(N)$ the sum of divisors of N.

N is said to be perfect if $\sigma(N) = 2N$.

It is one of the most infamous unsolved problems whether an odd perfect number exists.

It has been known an odd perfect number must satisfy various conditions.

Suppose N is an odd perfect number.

Euler has shown that $N = p^{\alpha} q_1^{2\beta_1} \cdots q_t^{2\beta_t}$ for distinct odd primes $p, q_1, \cdots, q_t$ with $p \equiv \alpha \equiv 1 \pmod{4}$.

In this talk, we would like to talk about $(\beta_1, \cdots, \beta_t)$.

There are known some results concerning forbidden values of $(\beta_1, \cdots, \beta_t)$.

- $(\beta_1, \cdots, \beta_t) \neq (1, \cdots, 1)$(Steuerwald, 1937).

- We cannot have $\beta_1 \equiv \cdots \equiv \beta_t \equiv 1 \pmod 3$(McDaniel, 1970).

If $\beta_1 = \cdots = \beta_t = \beta$, then it is known that

- $\beta \neq 2$(Kanold 1941),

- $\beta \neq 3$(Hagis and McDaniel 1972),

- $\beta \neq 5, 12, 17, 24, 62$(McDaniel and Hagis 1975),

- $\beta \neq 6, 8, 11, 14, 18$(Cohen and Williams 1985).

- $N \leq 2^{4^{4\beta^2+2\beta+3}}$(Yamada 2005).

McDaniel and Hagis conjecture that $\beta_1 = \cdots = \beta_t = \beta$ does not occur.

Now we have a question: can we have $\beta_1, \cdots, \beta_t \leq 2$? It is known that

- $(\beta_1, \beta_2, \cdots, \beta_t) \neq (2, 1, \cdots, 1)$(Kanold 1942, Brauer 1943).

- $(\beta_1, \beta_2, \cdots, \beta_t) \neq (2, 2, 1, \cdots, 1)$(Kanold 1953).

It is known that if $\beta_1, \cdots, \beta_t \leq 2$, then

- N does not have prime factor smaller than 739(Cohen 1987).

- $\alpha \neq 5$(Kanold 1953).

- $\alpha \equiv 1 \pmod{12}$ or $9 \pmod{12}$(McDaniel 1970).

Our results

Theorem 1. Let $N = q_0^\alpha q_1^2 \cdots q_s^2 q_{s+1}^4 \cdots q_{s+t}^4$ be an odd perfect number, where $q_0, q_1, \cdots, q_t$ are distinct primes, then N has a prime factor less than $\exp(4.97401 \times 10^{10})$.

Theorem 2. Let $N = q_0^{\alpha} q_1^2 \cdots q_s^2 q_{s+1}^4 \cdots q_{s+t}^4$ be an odd perfect number, where $q_0, q_1, \cdots, q_t$ are distinct primes, then N does not have prime factor smaller than 2500000.

Proof of Theorem 1

If $p \neq q_0$, then $q_i^2 + q_i + 1 \equiv 0 \pmod{p}$ for at most five primes q_i with $1 \le i \le s$. Similarly, $q_i^4 + q_i^3 + q_i^2 + q_i + 1 \equiv 0 \pmod{p}$ for at most five primes q_i with $s + 1 \le i \le s + t$.

It follows from classical sieve theory that the number of primes $\le x$ dividing N is $O(x/(\log x)^2)$ with an absolute implied constant.

So we can conclude that if $q_1, \cdots, q_{s+t} > C$, then $\sigma(N)/N < 2$. That is, if $N = p^e q_1^2 \cdots q_s^2 q_{s+1}^4 \cdots q_{s+t}^4$ is perfect, then N has a prime factor smaller than an effective computable constant C.

Computation of C requires a quantitative upper bound sieve(Greaves 2001) and various informations concerning the distribution of prime numbers (Rosser and Schoenfeld 1962, Ramare and Rumely 1996, Dusart 2001).

Proof of Theorem 2

Our idea is simple; for each prime $739 \leq p < 2500000$, we derive contradiction from the assumption that p is the smallest prime factor of N.

Our algorithm to derive contradiction is also simple.

1. Begin with $p_0 = p$.

2. For a prime p_i, we choose an integer $e_i \in \{1, 2, 4\}(\{2, 4\}$ if $e_j = 1$ for some $j < i)$.

3. If the smallest prime factor of $\sigma(p_i^{e_i})$ is smaller than p, then we return "error" for (p_i, e_i). Otherwise, choose a prime factor r of $\sigma(p_i^{e_i})$ different from $p_1, \cdots, p_i$, let $p_{i+1} = r$ and go to Step 1 with i incremented. If r yields "error", then we return "error" for (p_i, e_i).

4. If any choice e_i yields "error", then we return "error" for p_i. We go to Step 1 with i decremented.

In our implimentation, we take r as the smallest prime factor of $\sigma(p_i^{e_i})$ in Step 2 for simplicity. We implemented the procedure in C, using PARI-GP library for the large integer arithmetic. We executed our procedure in Celeron(R) 2.00GHz.

The procedure terminated for all primes ≤ 2500000 except 964697, 1121693, 1485413, 1666177, 1867003.

The procedure ran for

1. 21104 seconds for primes ≤ 964693,

2. 2811 seconds for primes ≥ 964703 and ≤ 1121689,

3. 16975 seconds for primes ≥ 1121699 and ≤ 1485397,

4. 1428 seconds for primes ≥ 1485433 and ≤ 1666151,

5. 1590 seconds for primes ≥ 1666201 and ≤ 1867001,

6. 2098 seconds for primes ≥ 1867009 and ≤ 2133277,

7. 96880 seconds for primes ≥ 2133281 and ≤ 2500000.

The most time-taking prime is 2133281, which required 71945 seconds. This is due to the prime factoring of $\sigma(\sigma(\sigma(2133281^2)^4)^2)$, a 102 digit number. This 102 digit number has a prime factor congruent to 31 (mod 60) and therefore unacceptable.

The largest value of i appearing in our procedure(we call the *depth*) was 4. The first prime with depth 4 is 30803.

We showed that neither of 964697, 1121693, 1485413, 1666177, 1867003 can be the smallest prime divisor of N by modifying a choice of r in Step 2.

This proves Theorem 2.

Unsolved problems

Despite of our effort, it is still unsolved whether there exists an odd sixth-power-free perfect number.

We pose some other related problems.

- Does our procedure always terminate?

We conjecture that our procedure terminates for any prime returing "error". If so, then an odd perfect number must be divisible by a sixth power of prime.

In view of Theorem 1, if our procedure terminates returning "error" for all primes below $\exp(4.97401 \times 10^{10})$, then we would show that an odd perfect number must be divisible by a sixth power of prime!

- Is there a prime with arbitrarily large depth?

We conjecture that although our procedure terminates for any prime, there exists a prime with arbitrarily large depth. Of course, it suffices to check all primes below $\exp(4.97401 \times 10^{10})$ and therefore the depth is bounded from a "practical" view!

- Are there infinitely many primes p for which $\sigma(p^2)$(or $\sigma(p^4)$) has no prime factor $\equiv 1 \pmod{15}$?

If $\sigma(p^2)$(or $\sigma(p^4)$) has a prime factor $q \equiv 1 \pmod{15}$, then $3 \mid \sigma(q^2)$ and $5 \mid \sigma(q^4)$. So q does not divide N unless $q = q_0$.

We conjecture that there are infinitely many primes p for which $\sigma(p^2)$(or $\sigma(p^4)$) has no prime factor $\equiv 1 \pmod{15}$.

This may be easier to prove than the conjecture that there are infinitely many primes p for which $\sigma(p^2)$(or $\sigma(p^4)$) is prime again.

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Thank you very much for listening!

THE END

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