Independence results for pattern sequences in different bases

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1 Introduction

Let $q \in \mathbb{Z}_{\geq 2}$**. Then** ${}^{\forall}n \in \mathbb{N}$ **is uniquely expressed as**

$$n = a_l q^l + a_{l-1} q^{l-1} + \cdots + a_0, \quad a_l > 0,$$

where $a_i \in \{0, 1, \ldots, q-1\}$.

The string of digits

$$(n)_q := a_l a_{l-1} \cdots a_1 a_0$$

is called the q**-***ary expansion* **of** n**.**

Definition 1.1

Σ_q^* : **set of all finite strings of elements in** $\{0, 1, \ldots, q-1\}$.

- $0, 1, 101, 1111 \in \Sigma_2^*$.
- $(n)_q \in \Sigma_q^*$ **for** ${}^{\forall}n \in \mathbb{N}$.

Let $m \in \mathbb{N}$ with $(m)_q = a_l a_{l-1} \cdots a_0$.

Definition 1.2

For $w \in \Sigma_q^*$ $(w \neq 0^s)$ with $|w| = k$, we define

$$e_q(w; m) := \sharp\{\, i \mid w = a_{i+k-1} \cdots a_i \}, \qquad (*)$$

$$e_q(w; 0) := 0.$$

$(*)$ **We assume that the q-ary expansion of n starts with an arbitrarily long string of zeros.**

Example

$e_{10}(1\,;\ 121333) = 2.$

$e_{10}(01\,;\ 121333) = 1.$

$e_{10}(33\,;\ 121333) = 2.$

$e_2(1;m)$: *sum of digit function*

$$e_2(1\,;\ 2^n - 1) = n,$$

$$\textit{since } (2^n - 1)_2 = \overbrace{11\cdots11}^{n}.$$

Definition 1.3

The sequence

$$\{e_q(w;n)\}_{n\geq 0}$$

is called the *pattern sequence* **for** $w \in \Sigma_q^*$.

Example

$$\{e_{10}(1;n)\}_{n\geq 0} = \{0,1,0,0,0,0,0,0,0,0,\\ 1,2,1,1,1,1,1,1,1,1,\\ 0,1,0,0,0,0,0,0,0,0,\\ \vdots\\ 0,1,0,0,0,0,0,0,0,0,\\ 1,2,1,1,1,1,1,1,1,1,\\ 2,3,2,2,2,2,2,2,2,2,\dots\}.$$

$$\{e_2(1;n)\}_{n\geq 0} = \{0,1,1,2,1,2,2,3,1,2,\\ 2,3,2,3,3,4,1,2,2,3,\dots\}.$$

We study the properties of the pattern sequence $\{e_q(\omega;n)\}_{n\geq 0}$ through their generating function

$$f(z) = \sum_{n\geq 0} e_q(\omega;n) z^n, \quad |z| < 1.$$

Theorem(Uchida, 1996)

For $^{\forall}q \geq 2$**,** $^{\forall}w \in \Sigma_q^*$**,**

$$f(z) = \sum_{n \geq 0} e_q(w;n) z^n$$

is transcendental over $\mathbb{C}(z)$**.**

$\Rightarrow$ $\{e_q(w;n)\}_{n \geq 0}$ **can not be a linear recurrence sequence!!**

Question

Let $q_1, q_2 \in \mathbb{Z}_{\geq 2}$ *with* $q_1 \neq q_2$**,**
$w_1 \in \Sigma_{q_1}^*$**,** $w_2 \in \Sigma_{q_2}^*$**.**
Can we find the relations between
$\{e_{q_1}(w_1;n)\}_{\geq 0}$ **and** $\{e_{q_2}(w_2;n)\}_{\geq 0}$**?**

2 Main results

Definition 2.1

$\zeta_1, \ldots, \zeta_m \in \mathbb{C}$: *algebraically dependent*

$$\Leftrightarrow {}^{\exists}P(z_1, \ldots, z_m) \in \mathbb{Z}[z_1, \ldots, z_m] \setminus \{0\}$$

such that $P(\zeta_1, \ldots, \zeta_m) = 0.$

If no such polynomial exists, then we say that the numbers $\zeta_1, \ldots, \zeta_m$ *are* *algebraically independent.*

Theorem 1.
Let $w_q \in \Sigma_q^*$ $(q = 2, 3 \ldots)$ *and*

$$f_q(z) = \sum_{n \geq 0} e_q(w_q; n) z^n, \quad q = 2, 3, \ldots.$$

Then their values

$$f_2(\alpha), f_3(\alpha), \ldots, f_m(\alpha), \ldots$$

are algebraically independent for any algebraic number α *with* $0 < |\alpha| < 1.$

Example

Let

$$f_1(z) = \sum_{n\ge 0} e_{10}(1;n)z^n, \qquad f_2(z) = \sum_{n\ge 0} e_2(1;n)z^n.$$

Then the values

$$f_1\left(\frac{1}{10}\right) = 0.100000000121111111\ldots,$$

$$f_2\left(\frac{1}{10}\right) = 0.1121223122323341223\ldots$$

are algebraically independent.

Corollary 1.

Let $w_q \in \Sigma_q^*$ $(q = 2, 3, \ldots, m)$. *Then the nontrivial linear combination over* $\mathbb{Q}$

$$\{c_1 e_2(w_2;n) + \cdots + c_{m-1} e_m(w_m;n)\}_{n\ge 0}$$

can not be a linear recurrence sequence.

Example

- $\{e_2(1;n)\}_{n\geq 0}$, $\{e_3(1;n)\}_{n\geq 0}, \ldots$ **are linearly independent over** $\mathbb{Q}$**.**

- $\{e_2(10;n)\}_{n\geq 0}$ **and** $\{e_4(1;n)\}_{n\geq 0}$ **which are defined by the number of 10's and 1's appearing in the dyadic and** 4**-ary expansion of** n**, respectively, are linearly independent over** $\mathbb{Q}$**.**

If w_s are chosen from a fixed set Σ_q^*, the corresponding generating functions can be algebraically dependent over $\mathbb{C}(z)$.

Example

Let $q = 2$, $w_1 = 01$, $w_2 = 10$. Then

$$\{e_2(01; n) - e_2(10; n)\}_{n \geq 0} = \{0, 1, 0, 1, \dots\}.$$

Hence, putting

$$f_1(z) = \sum_{n \geq 0} e_2(01; n) z^n,$$

$$f_2(z) = \sum_{n \geq 0} e_2(10; n) z^n,$$

we have

$$f_1(z) - f_2(z) = z + z^3 + z^5 + \cdots = \frac{z}{1 - z^2}.$$