A four-class association scheme derived from
a hyperbolic quadric in $PG(3, q)$

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Abstract

We prove the existence of a four-class association scheme on the set of external lines with respect to a hyperbolic quadric of $PG(3, q)$ where $q \geq 4$ is a power of 2. This result is an analogoue of the one by Ebert, Egner, Hollmann and Xiang. Taking a quotient of this association scheme yields a strongly regular graph of Latin square type. We show that this strongly regular graph can also be obtained by a generalization of the construction given by Mathon.

1 Introduction

In the paper [?], Ebert, Egner, Hollmann and Xiang constructed a four-class symmetric association scheme by using the set of secant lines with respect to an ovoid $\mathcal{O}$ of $PG(3, q)$ for $q \geq 4$ a power of 2. We can regard this association scheme as defined on the set of external lines by taking the null polarity with respect to $\mathcal{O}$. In this paper, we consider an analogous construction by a hyperbolic quadric. We construct a four-class symmetric association scheme by using the set of external lines with respect to a hyperbolic quadric of $PG(3, q)$. Each relation is invariant under the action of the orthogonal group $O^+(4, q)$ but the set of relations is not the set of orbitals on the set of external lines. Indeed, there are more orbitals than relations. Moreover, a quotient of this association scheme forms a strongly regular graph of Latin square type. We also prove that this strongly regular graph is isomorphic to the one constructed from a direct product of a pseudo-cyclic symmetric association scheme defined by the action of $SL(2, q)$ on the right cosets $SL(2, q)/O^-(2, q)$, which is a generalization of the construction given by Mathon [?]. This isomorphism is obtained by an isomorphism between $SL(2, q)^2$ and $\Omega^+(4, q)$.

2 Association schemes, strongly regular graphs and projective spaces

Let X be a finite set and let $\{R_i\}_{0 \leq i \leq d}$ be relations on X, that is, subsets of $X \times X$. Then $\mathfrak{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is called a *d-class symmetric association scheme* if the following conditions are satisfied.

1. $\{R_i\}_{0 \leq i \leq d}$ is a partition of $X \times X$.

2. R_0 is diagonal, that is, $R_0 = \{(x, x) \mid x \in X\}$.

3. $\{(y,x) \mid (x,y) \in R_i\} = R_i$ for any i.

4. For any $i,j,k \in \{0,1,\ldots,d\}$, $p_{ij}^k := |\{z \in X \mid (x,z) \in R_i, (y,z) \in R_j\}|$ is independent of the choice of (x,y) in R_k.

For $i \in \{0,\ldots,d\}$, let A_i be the adjacency matrix of the relation R_i, that is, A_i is indexed by X and

$$(A_i)_{xy} := \begin{cases} 1 & \text{if} \quad (x,y) \in R_i, \\ 0 & \text{if} \quad (x,y) \notin R_i. \end{cases}$$

Then we have

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

for any $i,j \in \{0,\ldots,d\}$. So $A_0, A_1, \cdots, A_d$ form a basis of the commutative algebra generated by $A_0, A_1, \cdots, A_d$ over the complex field (which is called the Bose-Mesner algebra of $\mathfrak{X}$). Moreover this algebra has a unique basis $E_0, E_1, \cdots, E_d$ of primitive idempotents. One of the primitive idempotents is $|X|^{-1}J$ where J is the matrix whose entries are all 1. So we may assume $E_0 = |X|^{-1}J$. Let $P = (p_j(i))_{0 \le i,j \le d}$ be the matrix defined by

$$(A_0 \ A_1 \ \cdots \ A_d) = (E_0 \ E_1 \ \cdots \ E_d)P.$$

We call P the first eigenmatrix of $\mathfrak{X}$. Remark that $\{p_j(i) \mid 0 \le i \le d\}$ is the set of eigenvalues of A_j. The first eigenmatrix satisfies the orthogonality relation:

$$\sum_{\nu=0}^{d} \frac{1}{k_\nu} p_\nu(i) p_\nu(j) = \frac{|X|}{m_i} \delta_{ij},$$

where $k_i = p_{ii}^0$ and $m_i = \text{rank } E_i$. We say that $\mathfrak{X}$ is *pseudo-cyclic* if there exists an integer m such that rank $E_i = m$ for all $i \in \{1,\cdots,d\}$. Remark that in this case, $|X| = dm+1$ and $k_i = p_{ii}^0 = m$ for all $i \in \{1,\cdots,d\}$ (see [**?**, p.76]).

Let G be a finite group and K be a subgroup of G. Then G acts naturally on the set $G/K \times G/K$ with orbitals $R_0, R_1, \ldots, R_d$, where we let $R_0 = \{(gK, gK) \mid gK \in G/K\}$. If all orbitals are self-paired, then $\mathfrak{X} = (G/K, \{R_i\}_{0 \le i \le d})$ forms a symmetric association scheme. We denote this association scheme by $\mathfrak{X}(G,K)$.

For a strongly regular graph with parameters (n,k,λ,μ), one of the eigenvalues of its adjacency matrix is k, and the others θ_1, θ_2 are the solutions of $x^2 + (\mu-\lambda)x + (\mu-k) = 0$. We can identify the pair of a strongly regular graph and its complement with a two-class symmetric association scheme whose first eigenmatrix is

$$\begin{bmatrix} 1 & k & n-k-1 \\ 1 & \theta_1 & -1-\theta_1 \\ 1 & \theta_2 & -1-\theta_2 \end{bmatrix} \tag{1}$$

In the paper [**?**], Mathon constructed a strongly regular graph from the pseudo-cyclic symmetric association scheme $\mathfrak{X}(SL(2,8), O^-(2,8))$. The next lemma is a generalization of this construction, due to Brouwer and Mathon [**?**]. Godsil [**?**] remarks that it can also be proved by Koppinen's identity [**?**] (see also [**?**, Theorem 2.4.1]).

Lemma 2.1. *Let $\mathfrak{X} = (X, \{R_i\}_{0\le i\le d})$ be a pseudo-cyclic symmetric association scheme on $dm+1$ points. Then the graph $\Delta(\mathfrak{X})$ whose vertex set is $X\times X$, where two distinct vertices (x,y) and (x',y') are adjacent if and only if $(x,x'), (y,y') \in R_i$ for some $i\neq 0$, is a strongly regular graph of Latin square type with parameters*

$$\left(|X|^2, m(|X|-1), |X|+m(m-3), m(m-1)\right).$$

Proof. The direct product of $\mathfrak{X}$ is $(X\times X, \{R_{ij}\}_{0\le i,j\le d})$, where

$$R_{ij} := \left\{\left((x,y),(x',y')\right) \mid (x,x')\in R_i,\ (y,y')\in R_j\right\}.$$

If P is the first eigenmatrix of $\mathfrak{X}$, then $P\otimes P$ is the first eigenmatrix of $(X\times X, \{R_{ij}\}_{0\le i,j\le d})$. The edge set of $\Delta(\mathfrak{X})$ is defined to be $\bigcup_{j=1}^{d} R_{jj}$. Then the eigenvalues of the adjacency matrix of $\Delta(\mathfrak{X})$ are

$$\left\{\sum_{j=1}^{d} p_j(i)p_j(i') \;\middle|\; 0\le i,i'\le d\right\}.$$

Since $\mathfrak{X}$ is psuedo-cyclic, $k_0 = m_0 = 1,\ k_j = m_i = m$ for $i,j\neq 0$. Hence the orthogonality relation implies

$$\sum_{j=1}^{d} p_j(i)p_j(i') = \frac{m|X|}{m_i}\delta ii' - m = \begin{cases} m(|X|-1) & \text{if } i=i'=0,\\ |X|-m & \text{if } i=i'\neq 0,\\ -m & \text{if } i\neq i'.\end{cases}$$

Therefore $\Delta(\mathfrak{X})$ has three eigenvalues. This implies that $\Delta(\mathfrak{X})$ is strongly regular. The parameters of $\Delta(\mathfrak{X})$ can easily be calculated. □

In Lemma **??**, if $\mathfrak{X} = \mathfrak{X}(G,K)$ for some finite group G and its subgroup K, then $\Delta(\mathfrak{X})$ has the following geometric interpretation.

Lemma 2.2. *Suppose that a finite group G and its subgroup K form a pseudo-cyclic symmetric association scheme $\mathfrak{X} = \mathfrak{X}(G,K)$. Then the graph $\Delta(\mathfrak{X})$ of Lemma **??** is isomorphic to the collinearity graph of the coset geometry $(G^2/K^2, G^2/D(G), *)$ where $D(G) := \{(x,x)\mid x\in G\}$ and for $x_1,x_2,y_1,y_2\in G$, $(x_1,x_2)K^2 * (y_1,y_2)D(G)$ if and only if $(x_1,x_2)K^2 \cap (y_1,y_2)D(G) = \emptyset$.*

Proof. Since each relation of $\mathfrak{X}(G,K)$ is an orbital of the action of G on $G/K\times G/K$, two pairs $(x_1K, y_1K), (x_2K, y_2K)$ are adjacent in the graph $\Delta(\mathfrak{X}(G,K))$ if and only if there exists $w\in G$ such that $y_1K = wx_1K, y_2K = wx_2K$. On the other hand, two pairs $(x_1,y_1)K^2, (x_2,y_2)K^2$ are adjacent in the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ if and only if $(x_1^{-1}x_2, y_1^{-1}y_2)$ is in $K^2D(G)K^2$ (cf. [**?**, p.15]).

For $x_1,x_2,y_1,y_2\in G$,

$$\begin{aligned}(x_1^{-1}x_2, y_1^{-1}y_2)\in K^2D(G)K^2 \quad&\Leftrightarrow\quad x_1^{-1}x_2,\ y_1^{-1}y_2\in KwK \text{ for some } w\in G,\\ &\Leftrightarrow\quad x_1^{-1}x_2\in Ky_1^{-1}y_2K\\ &\Leftrightarrow\quad y_1kx_1^{-1} = y_2k'x_2^{-1} \text{ for some } k,k'\in K,\\ &\Leftrightarrow\quad y_1\in wx_1K,\ y_2\in wx_2K \text{ for some } w\in G,\end{aligned}$$

Hence the mapping $G/K\times G/K \ni (xK,yK)\mapsto (x,y)K^2\in G^2/K^2$ is an isomorphism between the above two graphs. □

For the rest of this section, we recall some terminologies on finite projective spaces. In this paper, let q be a power of 2 and let $PG(3,q)$ be the three-dimensional projective space over $GF(q)$. For a non-degenerate quadratic form Q on $GF(q)^4$, we say that a point $p = \langle v \rangle$ is *singular* if $Q(v) = 0$, and we call the set of singular points a *quadric*. For a set of points X, we say that a line l is *external* (resp. *secant*) to the set X if the number of points in $l \cap X$ is 0 (resp. 2).

It is well known that there are two types of non-degenerate quadratic forms on $GF(q)^4$, which are called elliptic type or hyperbolic type. For a point p, denote by $p^\perp$ the orthogonal complement of p with respect to the symmetric bilinear form obtained from Q. Define for a line l or a plane π, $l^\perp := \bigcap_{p \in l} p^\perp$, $\pi^\perp := \bigcap_{p \in \pi} p^\perp$.

For a hyperbolic quadric in $PG(3,q)$, since q is even, the polarity $\perp$ is a null polarity, that is, if p is a point, then $p \in p^\perp$. More precisely, if p is on the hyperbolic quadric, then $p^\perp$ is the plane determined by the two generators of the quadric through p. If p is not on the quadric, then through p there are $q+1$ tangent lines to the quadric and these $q+1$ lines are coplaner. The plane determined by these $q+1$ tangent lines is $p^\perp$. For a line l, we have $l^\perp = \{\pi^\perp \mid l \subseteq \pi\}$. If l is external, then since every plane π containing l satisfies the point $\pi^\perp$ is nonsingular and not on l, the line $l^\perp$ is also external to the quadric and skew to l. On the other hand, for an external line l to an ovoid, the line $l^\perp$ is skew to l and secant to the ovoid (see [**?**, pp.24–26]).

A canonical form of quadratic form of hyperbolic type is

$$Q(x_1, x_2, x_3, x_4) = x_1 x_4 + x_2 x_3.$$

Denote by $\Omega^+(4,q)$ the commutator group of the orthogonal group defined from the above Q.

3 Main results

A four-class symmetric association scheme on the set of secant lines with respect to any ovoid was constructed:

Theorem 3.1. ([**?**]) *Let $q = 2^f \geq 4$. Then the following relations on the set of secant lines of $PG(3,q)$ with respect to an ovoid:*

$$\begin{aligned} R_1 &= \{(l,m) \mid l \cap m : \textit{a singular point}\ \} \\ R_2 &= \{(l,m) \mid l \cap m : \textit{a nonsingular point}\ \} \\ R_3 &= \left\{(l,m) \mid l^\perp \cap m \neq \emptyset\right\} \\ R_4 &= \left\{(l,m) \mid l \cap m = \emptyset,\ l^\perp \cap m = \emptyset\right\} \end{aligned}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

We can regard the above association scheme as defined on the set of external lines. The relations R_1, R_2, R_3 and R_4 correspond to the following relations on the set of external lines

$$\left\{(l,m) \;\middle|\; \langle l,m\rangle^{\perp}:\ \text{a singular point.}\right\},$$
$$\left\{(l,m) \;\middle|\; \langle l,m\rangle^{\perp}:\ \text{a nonsingular point.}\right\},$$
$$\left\{(l,m) \mid l^{\perp}\cap m \neq \emptyset\right\},$$
$$\left\{(l,m) \mid l\cap m = \emptyset,\ l^{\perp}\cap m = \emptyset\right\},$$

respectively.

In the paper [**?**], a plane π is called tangent if its orthogonal complement is a singular point.

For a hyperbolic quadric, we can construct a four-class symmetric association scheme similar to the above one. Let $\mathbf{L}$ be the set of external lines with respect to a hyperbolic quadric in $PG(3,q)$.

Theorem 3.2. *Let $q = 2^f \geq 4$. Then the following relations on the set $\mathbf{L}$ of external lines of $PG(3,q)$ with respect to a hyperbolic quadric:*

$$\begin{aligned} R_1 &= \{(l,m) \mid l\cap m : \textit{a point}\ \} \\ R_2 &= \left\{(l,m) \mid m = l^{\perp}\right\} \\ R_3 &= \left\{(l,m) \mid l^{\perp}\cap m : \textit{a point}\ \right\} \\ R_4 &= \left\{(l,m) \mid l\cap m = \emptyset,\ l^{\perp}\cap m = \emptyset\right\} \end{aligned}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

Moreover we can construct a strongly regular graph from this symmetric association scheme by taking a quotient.

Theorem 3.3. *Let $\Gamma = \Gamma_q$ ($q = 2^f \geq 4$) be the graph with vertex set $\{\{l,l^{\perp}\} \mid l \in \mathbf{L}\}$, where two distinct vertices of Γ, $\{l,l^{\perp}\}$, $\{m,m^{\perp}\}$ are adjacent if and only if $l\cap m \neq \emptyset$ or $l\cap m^{\perp} \neq \emptyset$. Then Γ is a strongly regular graph of Latin square type with parameters*

$$v = \frac{1}{4}q^2(q-1)^2,\ k = \frac{1}{2}(q-2)(q+1)^2,\ \lambda = \frac{1}{2}(3q^2-3q-4),\ \mu = q(q+1).$$

Remark that $l\cap m \neq \emptyset$ is equivalent to $l^{\perp}\cap m^{\perp} \neq \emptyset$, and $l\cap m^{\perp} \neq \emptyset$ is equivalent to $l^{\perp}\cap m \neq \emptyset$. So the adjacency in Γ is well-defined.

4 Proof of Theorem ??

To prove Theorem **??**, we recall some facts about $PG(3,q)$ with a hyperbolic quadric from Hirschfeld's book [**?**, §15–III]. From now on, put $q = 2^f \geq 4$. Let Π be the set of planes whose orthogonal complement is a nonsingular point.

Proposition 4.1. *For a hyperbolic quadric in $PG(3,q)$, the following statements hold.*
(i) *A plane containing an external line is in Π.*

(ii) *The number of external lines is* $q^2(q-1)^2/2$ *and there are* $q+1$ *planes of* Π *containing a given external line.*
(iii) *The number of planes in* Π *is* $q(q^2-1)$ *and there are* $q(q-1)/2$ *external lines in a given plane of* Π.
(iv) *For* $\pi \in \Pi$, *there is no external line through* $\pi^\perp$ *on* π. *For a nonsingular point* p *of* π *distinct from* $\pi^\perp$, *there are* $q/2$ *external lines through* p *on* π.
(v) *There are* $q(q-1)/2$ *external lines through a given nonsingular point.*

(Remark: when q is an odd prime power, (i),(ii),(iii) and (v) also hold. For a plane π of Π, $\pi^\perp$ is not in π.)

First we show that the relations $R_0, \dots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$. It is clear that any pair (l,m) of $\mathbf{L} \times \mathbf{L}$ is in one of $\{R_i\}_{0 \le i \le 4}$. Since any external line l is skew to $l^\perp$, R_1 and R_2 have no intersection. Suppose that $l, m \in \mathbf{L}$ satisfy that l meets m. Then the point $\langle l, m\rangle^\perp$ is on $l^\perp$, so m is skew to $l^\perp$ by Proposition **??** (iv). Hence R_1 and R_3 have no intersection. Therefore $\{R_i\}_{0 \le i \le 4}$ is a partition of $\mathbf{L} \times \mathbf{L}$.

Next we show that each relation is symmetric. It is clear that R_1, R_2 and R_4 are symmetric. If $(l,m) \in R_3$, then $\langle l^\perp, m\rangle$ forms a plane and $\langle l^\perp, m\rangle^\perp = l \cap m^\perp$, hence $(m,l) \in R_3$. Therefore R_3 is also symmetric.

Finally we show that for any $i,j,k \in \{0, \cdots, 4\}$,

$$p_{ij}^k = |\{n \in \mathbf{L} \mid (l,n) \in R_i, (n,m) \in R_j\}|$$

is independent of the choice of $(l,m) \in R_k$. The assertion is clear when $k=0$. For the moment, we put $p_{ij}(l,m) = |\{n \in \mathbf{L} \mid (l,n) \in R_i, (n,m) \in R_j\}|$. We can easily see that when $(l,m) \in R_k$, $p_{0j}(l,m) = \delta_{jk}$. Since each relation is symmetric, $p_{ji}(l,m) = p_{ij}(m,l)$. Since $R_0, \dots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$, we have

$$\sum_{i=0}^{4} p_{ii}^0 = |\mathbf{L}| = \frac{1}{2}q^2(q-1)^2,$$

and

$$\sum_{j=0}^{4} p_{ij}(l,m) = p_{ii}^0$$

for any $i \in \{0, \dots, 4\}$ and for any pair (l,m). Let σ be the permutation $(0,2)(1,3)$ on $\{0, \cdots, 4\}$. Then since $(l,m) \in R_i$ if and only if $(l, m^\perp) \in R_{\sigma(i)}$,

$$p_{ij}(l,m) = p_{i\sigma(j)}(l, m^\perp) = p_{\sigma(i)\sigma(j)}(l,m). \tag{2}$$

Hence we only need to show that p_{11}^k $(1 \le k \le 4)$ are independent of the choice of $(l,m) \in R_k$.

Lemma 4.2. *For* $1 \le k \le 4$, p_{11}^k *is independent of the choice of* $(l,m) \in R_k$ *and*

$$p_{11}^0 = \frac{1}{2}(q-2)(q+1)^2,\ p_{11}^1 = q^2 - \frac{3}{2}q - 2,\ p_{11}^2 = 0,\ p_{11}^3 = \frac{1}{2}q^2,\ p_{11}^4 = \frac{1}{2}q(q+1).$$

Proof. Fix $l \in \mathbf{L}$. Any line which meets l in a point is in a plane through l, and conversely any line in a plane through l meets l in a point. Hence by Proposition **??** (ii) and (iii),

$$\begin{aligned} p^0_{11} &= |\{n \in \mathbf{L} \mid (l,n) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid n \subset \pi,\ n \neq l\}| \\ &= (q+1) \times \left(\frac{1}{2}q(q-1) - 1\right) \\ &= \frac{1}{2}(q-2)(q+1)^2 \end{aligned}$$

where $\Pi_l := \{\pi \in \Pi \mid l \subset \pi\}$.

For $(l,m) \in R_1$, if $n \in \mathbf{L}$ meets both l and m, then n has a point $l \cap m$ or n is in the plane $\langle l, m\rangle$. Hence by Proposition **??** (iii)–(v),

$$\begin{aligned} p^1_{11} &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= |\{n \in \mathbf{L} \mid n \subseteq \langle l,m\rangle,\ n \neq l,m\}| + |\{n \in \mathbf{L} \mid l \cap m \in n \not\subseteq \langle l,m\rangle\}| \\ &= \left(\frac{1}{2}q(q-1) - 2\right) + \left(\frac{1}{2}q(q-1) - \frac{1}{2}q\right) \\ &= q^2 - \frac{3}{2}q - 2. \end{aligned}$$

From (**??**), we have $p^2_{11} = 0$. For $(l,m) \in R_3 \cup R_4$, we have

$$|\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}|.$$

If $(l,m) \in R_3$, then there is just one plane $\pi_0 = \langle l, l^\perp \cap m\rangle \in \Pi_l$ such that $\pi_0^\perp = m \cap \pi_0$. By Proposition **??** (iv), there is no line of $\mathbf{L}$ through $m \cap \pi_0$ and in π_0, and for other plane π, there are $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$ and in π. Hence

$$\begin{aligned} p^3_{11} &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l \setminus \{\pi_0\}} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}| \\ &= q \times \frac{1}{2}q. \end{aligned}$$

For $(l,m) \in R_4$, any plane π of Π_l has $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$. So

$$p^4_{11} = |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = (q+1) \times \frac{1}{2}q.$$

□

Therefore $(\mathbf{L}, \{R_i\}_{0 \le i \le 4})$ becomes a symmetric association scheme. For $i \in \{0,\dots,d\}$, let $B_i := (p^k_{ij})_{0 \le j,k \le 4}$. Then B_0 is the identity matrix,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ p^0_{11} & q^2 - 3/2q - 2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p^0_{11} & q^2 - 3/2q - 2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \end{pmatrix},$$

where $s = (q+1)(q^2-3q-2)/2$,

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p_{11}^0 & q^2-3/2q-2 & q(q+1)/2 \\ 0 & 1 & 0 & 0 & 0 \\ p_{11}^0 & q^2-3/2q-2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \end{pmatrix},$$

and

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \\ 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \\ p_{44}^0 & q(q-3)(q^2-3q-2)/2 & p_{44}^0 & s & t \end{pmatrix},$$

where $p_{44}^0 = q(q-2)(q-3)(q+1)/2$ and $t = q(q-3)(q^2-3q-2)/2$. The first eigenmatrix of this association scheme is given by

$$P = \begin{pmatrix} 1 & (q-2)(q+1)^2/2 & 1 & (q-2)(q+1)^2/2 & p_{44}^0 \\ 1 & (q-2)(q+1)/2 & -1 & -(q-2)(q+1)/2 & 0 \\ 1 & -(q+1) & -1 & q+1 & 0 \\ 1 & -(q+1) & 1 & -(q+1) & 2q \\ 1 & (q^2-3q-2)/2 & 1 & (q^2-3q-2)/2 & -q(q-3) \end{pmatrix}.$$

5 Proof of Theorem ??

In this section, we prove Theorem **??** by using Theorem **??**. The number of vertices of the graph Γ is $|\mathbf{L}|/2 = q^2(q-1)^2/4$. For a pair $\{l, l^\perp\} \in V\Gamma$,

$$\begin{aligned} &\{\{m, m^\perp\} \in V\Gamma \mid \{m, m^\perp\} \text{ is adjacent to } \{l, l^\perp\}\} \\ &\quad = \{\{m, m^\perp\} \in V\Gamma \mid \ m \text{ meets } l \text{ in a point. }\} \\ &\quad = \{\{m, m^\perp\} \in V\Gamma \mid (l, m) \in R_1\}. \end{aligned}$$

So, the size of this set is $p_{11}^0 = (q-2)(q+1)^2/2$, which is just k in the definition of strongly regular graph. Next choose $\{l, l^\perp\}, \{m, m^\perp\} \in V\Gamma$ which are adjacent in Γ. We may suppose that l meets m in a point. Then

$$\begin{aligned} &\{\{n, n^\perp\} \in V\Gamma \mid \{n, n^\perp\} \text{ is adjacent to both } \{l, l^\perp\} \text{ and } \{m, m^\perp\}\} \\ &\quad = \{\{n, n^\perp\} \in V\Gamma \mid \ n \text{ meets both } l \text{ and } m \text{ in a point.}\} \\ &\qquad\quad \cup \{\{n, n^\perp\} \in V\Gamma \mid \ n \text{ meets both } l \text{ and } m^\perp \text{ in a point. }\}, \\ &\quad = \{\{n, n^\perp\} \in V\Gamma \mid (l, n) \in R_1,\ (m, n) \in R_1 \cup R_3\}. \end{aligned}$$

Hence the size of this set is $p_{11}^1 + p_{13}^1 = (3q^2-3q-4)/2$. This is just λ in the definition of strongly regular graph.

Similarly, for $\{l,l^\perp\},\{m,m^\perp\}\in V\Gamma$ which are not adjacent in Γ, since $(l,m)\in R_4$,

$$\begin{aligned}&\left|\left\{\{n,n^\perp\}\in V\Gamma \;\middle|\; \{n,n^\perp\} \text{ is adjacent to both } \{l,l^\perp\} \text{ and } \{m,m^\perp\}\right\}\right|\\ &= \; p^4_{11}+p^4_{13}=q(q+1).\end{aligned}$$

This is just μ in the definition of strongly regular graph.

Alternatively, we can prove Theorem **??** by using the quotient association scheme (cf. [**?**, p.139, Thm.9.4]). In the association scheme of Theorem 3.2, $R_0\cup R_2$ is an equivalence relation on $\mathbf{L}$. So we can define a quotient association scheme on the set of equivalence classes $\big\{\{l,l^\perp\}\mid l\in\mathbf{L}\big\}$ whose relations are

$$\begin{aligned}&\left\{(\{l,l^\perp\},\{m,m^\perp\})\mid (l,m)\in R_1\cup R_3\right\}=\text{the edge set of }\Gamma,\\ &\left\{(\{l,l^\perp\},\{m,m^\perp\})\mid (l,m)\in R_4\right\},\end{aligned}$$

and the diagonal relation. The first eigenmatrix of this association scheme can be computed from P (cf. [**?**, p.148]):

$$\begin{pmatrix} 1 & (q-2)(q+1)^2/2 & q(q-2)(q-3)(q+1)/4 \\ 1 & -(q+1) & q \\ 1 & (q^2-3q-2)/2 & -q(q-3)/2 \end{pmatrix}.$$

The first relation forms a strongly regular graph whose parameters are calculated from the second column of the above first eigenmatrix.

6 Another construction of Γ_q

In this section, we will give another construction of the strongly regular graph Γ_q. This construction uses a method which generalizes a construction of Mathon ([**?**, p.137], see also [**?**, pp.96–97]).

Let $G=SL(2,q)$, $K=O^-(2,q)$. Then $\mathfrak{X}(G,K)$ is a $(q-2)/2$-class pseudo-cyclic symmetric association scheme (cf. [**?**, p.96]). By Lemma **??**, we can construct a strongly regular graph $\Delta(\mathfrak{X}(G,K))$ with parameters

$$\left(\frac{1}{4}q^2(q-1)^2,\ \frac{1}{2}(q-2)(q+1)^2,\ \frac{1}{2}(3q^2-3q-4),\ q(q+1)\right)$$

which are the same as those of Γ_q. We shall prove that these graphs are isomorphic.

To show this, we use the isomorphism $G^2\simeq\Omega^+(4,q)$ which maps (X,Y) to $X\otimes Y$ (see [**?**, p.199]). Let l_0 be the external line generated by $v_1={}^t(0,1,1,0), v_2={}^t(1,1,0,\alpha)$, where α is an element of $\mathbf{F}_q$ such that the polynomial $x^2+x+\alpha$ is irreducible over $\mathbf{F}_q$. For an external line l, there are $2(q+1)$ basis (u_1,u_2) of l such that $Q(xu_1+yu_2)=x^2+xy+\alpha y^2$ for any $x,y\in\mathbf{F}_q$. Indeed, by Witt's Theorem, K acts regularly on the set of basis (u_1,u_2) of l with the above condition. It follows that the size of this set is equal to $|K|=2(q+1)$. Let $\mathcal{P}$ be the set of nonsingular points in $PG(3,q)$ and let $\mathcal{L}=\{l\cup l^\perp\mid l\in\mathbf{L}\}$. Then the following lemma holds.

Lemma 6.1. *The group $\Omega^+(4,q)=\{X\otimes Y\mid X,Y\in G\}$ is flag-transitive on the incidence structure $(\mathcal{P},\mathcal{L},\in)$. Under the isomorphism $G^2\simeq\Omega^+(4,q)$, the groups $D(G)$, K^2 are the stabilizers of an element of $\mathcal{P}$, $\mathcal{L}$, respectively.*

Proof. Let $X=(x_{ij})_{1\le i,j\le 2},\ Y=(y_{ij})_{1\le i,j\le 2}\in G$. Since

$$X\otimes Y=\left(\begin{array}{cccc} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12}\\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22}\\ x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12}\\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22}\end{array}\right),$$

$X\otimes Y$ fixes v_1 if and only if

$$\begin{array}{rcl} x_{11}y_{12}+x_{12}y_{11}=x_{21}y_{22}+x_{22}y_{21} & = & 0,\\ x_{11}y_{22}+x_{12}y_{21}=x_{21}y_{12}+x_{22}y_{11} & = & 1.\end{array}$$

This implies

$$\Omega^+(4,q)_{v_1}=\{X\otimes X\ |X\in G\}\simeq D(G). \tag{3}$$

For $X\in G$, $X\otimes X$ fixes v_2 if and only if

$$\begin{array}{rcl} x_{11}^2+x_{11}x_{12}+\alpha x_{12}^2 & = & 1,\\ x_{11}x_{21}+x_{12}x_{21}+\alpha x_{12}x_{22} & = & 0,\\ x_{21}^2+x_{21}x_{22}+\alpha x_{22}^2 & = & \alpha.\end{array}$$

From these, we have

$$\Omega^+(4,q)_{v_1,v_2}=\left\{X\otimes X\ \middle|\ X=\left(\begin{array}{cc} a & b\\ \alpha b & a+b\end{array}\right)\in G\right\}$$

which is of order $q+1$. Hence

$$\begin{array}{rcl} \left|\left\{(Mv_1,Mv_2)\mid M\in\Omega^+(4,q)\right\}\right| & = & |\Omega^+(4,q)|/(q+1)\\ & = & q^2(q-1)^2(q+1).\end{array}$$

Since $Q(xv_1+yv_2)=x^2+xy+\alpha y^2$ for any $x,y\in\mathbf{F}_q$,

$$\begin{array}{l} \left|\left\{(u_1,u_2)\ \middle|\ Q(xu_1+yu_2)=x^2+xy+\alpha y^2\quad {}^\forall x,y\in\mathbf{F}_q\right\}\right|\\ \quad=\ |\mathbf{L}|\times 2(q+1)\\ \quad=\ q^2(q-1)^2(q+1).\end{array}$$

Hence $\Omega^+(4,q)$ acts transitively on the set of pairs (u_1,u_2) such that $Q(xu_1+yu_2)=x^2+xy+\alpha y^2$ for any $x,y\in\mathbf{F}_q$. In particular, $\Omega^+(4,q)$ is flag-transitive on $(\mathcal{P},\mathcal{L},\in)$.

The equality (**??**) means that the stabilizer of $\langle v_1\rangle\in\mathcal{P}$ is isomorphic to $D(G)$. Let

$$A:=\left(\begin{array}{cc} 1 & 0\\ 1 & 1\end{array}\right),\ B:=\left(\begin{array}{cc} a_0 & b_0\\ \alpha b_0 & a_0+b_0\end{array}\right)\in G$$

such that B is of order $q+1$. Then the group $\langle A,B\rangle$ is isomorphic to K. $A\otimes I, I\otimes A$ interchange l_0 and $l_0^\perp$, while $B\otimes I, I\otimes B$ fix l_0 and $l_0^\perp$. So $\{X\otimes Y\mid X,Y\in\langle A,B\rangle\}$ is a subgroup of $\Omega^+(4,q)_{l_0\cup l_0^\perp}$. Since $\Omega^+(4,q)_{l_0\cup l_0^\perp}$ has order $4(q+1)^2=|K|^2$, we have that $\Omega^+(4,q)_{l_0\cup l_0^\perp}$ is isomorphic to K^2. ∎

Theorem 6.2. *The graph* Γ_q *is isomorphic to* $\Delta(\mathfrak{X}(SL(2,q), O^-(2,q)))$.

Proof. The graph Γ_q is isomorphic to the collinearity graph of the dual of the incidence structure $(\mathcal{P}, \mathcal{L}, \in)$. From Lemma **??**, the dual of $(\mathcal{P}, \mathcal{L}, \in)$ is isomorphic to the coset geometry $(G^2/K^2, G^2/D(G), *)$ defined in Lemma **??**. From Lemma **??**, the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ is isomorphic to $\Delta(\mathfrak{X}(G,K))$. Therefore Γ_q is isomorphic to $\Delta(\mathfrak{X}(G,K))$. $\square$

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