Standard pair decompositions of toric ideals and minimum cost flow problems

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Abstract

To integer programming problems, computational algebraic approaches using Gröbner bases or standard pairs via the discreteness of toric ideals have been studied in recent years. Although these approaches do not give improved time complexity bound compared with existing methods for solving integer programming problems, these give algebraic analysis for structure of integer programming problems. In this paper, we focus on the minimum cost flow problems, whose structure is well-known, and analyze using standard pairs. Especially, using some results about Gröbner bases for toric ideals and hypergeometric functions, we show that the number of vertices of the (nondegenerate) dual polyhedra for minimum cost flow problems on acyclic directed graphs is more than 1 and less than the Catalan number.

1 Introduction

Recently, some algebraic approaches to integer programming problems have been studied. The two main approaches are an approach using *Gröbner bases* [4] and that using *standard pair decompositions* [6]. Although they do not give improved complexity bounds compared with existing methods, these approaches themselves are very interesting by applying computational algebraic methods to such hard problems, and give algebraic analysis for structure of integer programming problems. Several results about structure of integer programming problems have givin using Gröbner bases [4, 11, 13, 14, 16] and standard pairs [6, 7].

The minimum cost flow problem forms a well-known subclass of integer programming problems which can be solved in polynomial time. Gröbner basis approach for the minimum cost flow problems is a variant of cycle-canceling algorithm, in the sense that, for any feasible flow, we can obtain the optimal flow by augmenting flows along the cycles which correspond to the elements of the Gröbner basis as many as possible. On the other hand, standard pair approach for the minimum cost flow problems first finds the set of standard pairs, and solves linear system of equations for each standard pair until an integer and non-negative solution is obtained.

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Since each standard pair for the minimum cost flow problem corresponds to a vertex of (non-degenerate) dual polyhedron of the problem, this approach is a variant of the enumeration of dual bases. For more details about enumerations of polyhedrons, see e.g. [15].

The fact that each standard pair corresponds to a vertex of dual polyhedron of the minimum cost flow problem also implies that the number of standard pairs for the minimum cost flow problem gives the number of vertices of dual polyhedron of the problem. The number of vertices of dual polyhedra for some special problems have been studied. Especially, for the transportation problem on $m \times n$ bipartite graph, which is equivalent to the minimum cost flow problem on the graph with $m+n$ vertices, the number of vertices is at most $\binom{m+n-2}{m-1}$ [2]. We show that, for the minimum cost flow problem on an *acyclic* directed graph with d vertices, the number of vertices of the dual polyhedron is at most $\frac{1}{d}\binom{2(d-1)}{d-1}$, which is the $(d-1)$-th Catalan number. This upper bound is shown by combining two results, one is about a characterization of Gröbner basis for minimum cost flow problem [8], and the other is about the special hypergeometric function [5].

This paper is organized as follows. In Section 2, standard pair decompositions are defined and their relations to regular triangulations, dual polyhedra and integer programming problems are introduced. In Section 3, we characterize the standard pairs for the minimum cost flow problems. In Section 4, the minimum and the maximum number of vertices of dual polyhedra are shown for the minimum cost flow problems on acyclic tournament graphs. The methods which are used to show these results are also shown. For the problems on acyclic directed graphs, these results give upper and lower bounds. Finally in Section 5 we conclude this paper.

2 Standard pair decompositions of integer programs

In this section, we give some definitions about standard pairs, and their relations to regular triangulations, dual polyhedra and integer programmings. We refer to [12] for the introduction of standard pairs, and [6, 7] for their applications.

2.1 Standard pairs, regular triangulations and dual polyhedra

We fix a matrix $A \in \mathbb{Z}^{d\times n}$ and a cost vector $\boldsymbol{c} \in \mathbb{R}^n$, and consider the family $IP_{A,\boldsymbol{c}}$ of integer programming problems $IP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize\{\boldsymbol{c}\cdot\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \in \mathbb{N}^n\}$ as $\boldsymbol{b}$ varies in $\{A\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^n\} \subseteq \mathbb{Z}^d$ ($\mathbb{N}$ is the set of non-negative integers, and $\mathbb{Z}$ is the set of integers). We assume that $\boldsymbol{c}$ is *generic* in the sense that each program in $IP_{A,\boldsymbol{c}}$ has a unique optimal solution. For a fixed $\boldsymbol{c}$, let $O_{\boldsymbol{c}} \subset \mathbb{N}^n$ be the set of all the optimal solutions to all programs in $IP_{A,\boldsymbol{c}}$, and $N_{\boldsymbol{c}} := \mathbb{N}^n \setminus O_{\boldsymbol{c}}$. Then there exists a unique minimal finite set $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_s\} \subset \mathbb{N}^n$ such that $N_{\boldsymbol{c}} = \bigcup_{i=1}^{s}(\boldsymbol{p}_i + \mathbb{N}^n)$, where $\boldsymbol{p}_i + \mathbb{N}^n := \{\boldsymbol{p}_i + \boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^n\}$ [13]. Algebraically, $\boldsymbol{p}_1, \dots, \boldsymbol{p}_s$ are the set of exponent vectors of initial terms of $G_{\boldsymbol{c}}$, which is the reduced Gröbner bases of the toric ideal I_A with respect to $\boldsymbol{c}$.

For $\boldsymbol{a} \in \mathbb{N}^n$ and $\sigma \subseteq \{1, \dots, n\}$, we define the set of points $(\boldsymbol{a}, \sigma)$ in $\mathbb{N}^n$ as $(\boldsymbol{a}, \sigma) := \left\{\boldsymbol{a} + \sum_{i\in\sigma} k_i\boldsymbol{e}_i \mid k_i \in \mathbb{N}\right\}$, where $\boldsymbol{e}_i$ is an i-th unit vector in $\mathbb{R}^n$.

Definition 2.1 *$(\boldsymbol{a}, \sigma)$ is a* standard pair *of $O_{\boldsymbol{c}}$ if*

(i) *the support of $\boldsymbol{a}$ is contained in $\{1, \dots, n\} \setminus \sigma$,*

(ii) *$(\boldsymbol{a}, \sigma) \subseteq O_{\boldsymbol{c}}$, and*

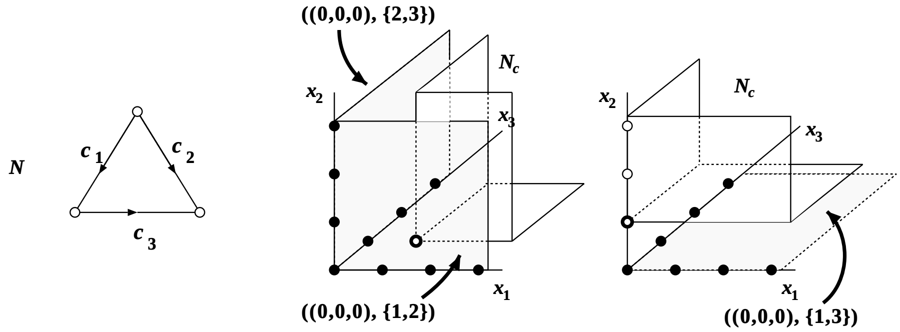

Figure 1: The network N and two types of standard pair decompositions

(iii) $(\boldsymbol{a}, \sigma) \not\subset (\boldsymbol{b}, \tau)$ *for any other* $(\boldsymbol{b}, \tau)$ *which satisfies* (i) *and* (ii).

We denote $S(O_{\boldsymbol{c}})$ *the set of all standard pairs of* $O_{\boldsymbol{c}}$.

The standard pairs of $O_{\boldsymbol{c}}$ induce a unique covering of $O_{\boldsymbol{c}}$, which we call the *standard pair decomposition* of $O_{\boldsymbol{c}}$. $|S(O_{\boldsymbol{c}})|$ is called the *arithmetic degree* of $O_{\boldsymbol{c}}$ and denoted by *arith-deg* $(O_{\boldsymbol{c}})$. Because of the genericity of $\boldsymbol{c}$, $O_{\boldsymbol{c}}$ is not empty and *arith-deg* $(O_{\boldsymbol{c}}) \geq 1$.

Example 2.2 *Let A be the matrix which is obtained from the incidence matrix of the network shown in Figure 1. For $\boldsymbol{c} = (c_1, c_2, c_3)$ with $c_1 + c_3 > c_2$, the standard pairs of $O_{\boldsymbol{c}}$ are* $((0,0,0), \{1,2\})$ *and* $((0,0,0), \{2,3\})$, *thus the arithmetic degree of* $O_{\boldsymbol{c}}$ *is* 2. *On the other hand, for $\boldsymbol{c}$ with $c_1 + c_3 < c_2$, the standard pair of $O_{\boldsymbol{c}}$ is* $((0,0,0), \{1,3\})$, *thus the arithmetic degree of* $O_{\boldsymbol{c}}$ *is* 1.

Let $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ be the column vectors of A and *cone*(A) the cone generated by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. For $\sigma \subset \{1, \ldots, n\}$, we denote A_σ for the submatrix of A whose columns are indexed by σ. For a cost vector $\boldsymbol{c}$, we define the *regular triangulation* $\Delta_{\boldsymbol{c}}$ of *cone*(A) as follows: *cone*(A_σ) is a face of $\Delta_{\boldsymbol{c}}$ if and only if there exists a vector $\boldsymbol{y} \in \mathbb{R}^d$ such that $\boldsymbol{y} \cdot \boldsymbol{a}_j = c_j \ (j \in \sigma)$ and $\boldsymbol{y} \cdot \boldsymbol{a}_j < c_j \ (j \notin \sigma)$. If *cone*$(A_\sigma)$ is a face of $\Delta_{\boldsymbol{c}}$, σ also is called a *face* of $\Delta_{\boldsymbol{c}}$. The genericity of $\boldsymbol{c}$ implies that $\Delta_{\boldsymbol{c}}$ is in fact a triangulation (i.e. each face of $\Delta_{\boldsymbol{c}}$ is simplicial) [11].

Lemma 2.3 ([10, 12])

(i) *If* $O_{\boldsymbol{c}}$ *has* $(*, \sigma)$ *as a standard pair, then* σ *is a face of* $\Delta_{\boldsymbol{c}}$.

(ii) $O_{\boldsymbol{c}}$ *has* $((0, \ldots, 0), \sigma)$ *as a standard pair if and only if* σ *is a maximal face of* $\Delta_{\boldsymbol{c}}$.

(iii) *If* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ *span an affine hyperplane, then* $\Delta_{\boldsymbol{c}}$ *is same as a regular triangulation of conv*(A) *with respect to* $\boldsymbol{c}$, *and the number of standard pairs* $(*, \sigma)$ *for a maximal face* σ *of* $\Delta_{\boldsymbol{c}}$ *equals the normalized volume of* σ *in* $\Delta_{\boldsymbol{c}}$.

When the vertices of *conv*(A) are in the m-dimensional lattice $L \simeq \mathbb{Z}^m$, we define the *normalized volume* of a maximal face σ of $\Delta_{\boldsymbol{c}}$ by the volume of σ with the normalization that the volume of the convex hull of $0, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_m$ is 1. Here, $\{\boldsymbol{e}_i\}_{1 \leq i \leq m}$ are the basis of the lattice L.

For a polyhedron $P \subset \mathbb{R}^n$ and a face F of P, the *normal cone* of F at P is the cone $N_P(F) := \{\omega \in \mathbb{R}^n \mid \omega \cdot x' \geq \omega \cdot x \text{ for all } x' \in F \text{ and } x \in P\}$. The set of normal cones for all faces of P is called the *normal fan* of P.

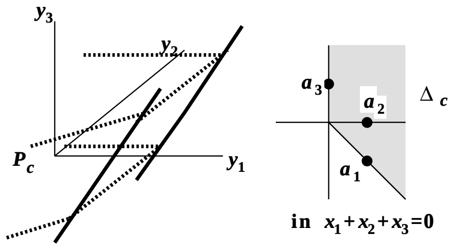

Figure 2: The dual polyhedron $P_{\boldsymbol{c}}$ and the triangulation $\Delta_{\boldsymbol{c}}$ of Example 2.2

Lemma 2.4 ([7]) *$\Delta_{\boldsymbol{c}}$ is the normal fan of the polyhedron $P_{\boldsymbol{c}} := \{\boldsymbol{y} \in \mathbb{R}^d \mid \boldsymbol{y}A \leq \boldsymbol{c}\}$.*

We remark that $P_{\boldsymbol{c}}$ is the dual polyhedron of the linear relaxation problem

$$LP_{A,\boldsymbol{c}}(\boldsymbol{b}) := minimize \{\boldsymbol{c} \cdot \boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

of $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. This lemma shows that there is one-to-one correspondence between the minimal faces of $P_{\boldsymbol{c}}$ and the maximal faces of $\Delta_{\boldsymbol{c}}$.

Example 2.2 (continued.) *Figure 2 shows the dual polyhedron $P_{\boldsymbol{c}}$ and $\Delta_{\boldsymbol{c}}$ for $\boldsymbol{c}$ with $c_1+c_3 > c_2$. We remark that cone(A) is contained in the hyperplane $x_1 + x_2 + x_3 = 0$.*

2.2 Standard pair decompositions and integer programs

Using the standard pair decomposition of $O_{\boldsymbol{c}}$, $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ can be solved by solving at most (arithmetic degree of $O_{\boldsymbol{c}}$)-many linear systems of equations [6]. We describe the outline of this approach.

Let $\boldsymbol{u}$ be the optimal solution to $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$. Since the standard pairs cover $O_{\boldsymbol{c}}$, $\boldsymbol{u}$ is covered by some standard pair $(\boldsymbol{a}, \sigma)$. Thus $\boldsymbol{u} = \boldsymbol{a} + \sum_{i\in\sigma} k_i\boldsymbol{e}_i$ for some non-negative integers $\{k_i\}_{i\in\sigma}$, and

$$\begin{aligned}\boldsymbol{b} &= A\boldsymbol{u} \\ &= A\left(\boldsymbol{a} + \sum_{i\in\sigma} k_i\boldsymbol{e}_i\right) \\ &= A\boldsymbol{a} + \sum_{i\in\sigma} k_i\boldsymbol{a}_i.\end{aligned}$$

Lemma 2.3 implies that $\{\boldsymbol{a}_i\}_{i\in\sigma}$ are linearly independent, and therefore $\{k_i\}_{i\in\sigma}$ is the unique solution to the linear system

$$\sum_{i\in\sigma} x_i\boldsymbol{a}_i = \boldsymbol{b} - A\boldsymbol{a}. \tag{1}$$

This observation induces an algorithm to solve $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ using the standard pair decomposition of $O_{\boldsymbol{c}}$.

Algorithm 2.3 (Solving $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ using $S(O_{\boldsymbol{c}})$)

(i) *For* $(\boldsymbol{a}, \sigma) \in S(O_{\boldsymbol{c}})$*, solve the linear system* (1)*. Let* $\{k_i\}_{i\in\sigma}$ *be the solution.*

(ii) *If* $\{k_i\}_{i\in\sigma}$ *are all both integral and non-negative, output* $\boldsymbol{a} + \sum_{i\in\sigma} k_i \boldsymbol{e}_i$ *as the optimal solution. Otherwise, repeat* (i) *for another standard pair.*

This algorithm solves at most *arith-deg* $(O_{\boldsymbol{c}})$-many linear systems of equations. Therefore arithmetic degree of $O_{\boldsymbol{c}}$ is a measure of the complexity of $IP_{A,\boldsymbol{c}}$.

3 Standard pairs of directed graphs

We consider that A is the incidence matrix of a network $N = (V, E)$ such that $|V| = d$, $|E| = n$, $\boldsymbol{b} \in \{A\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^n\}$ and $\boldsymbol{c} \in \mathbb{R}^n_{\geq 0}$. This means that $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is an uncapacitated minimum cost flow problem on N with the demand/supply vector $\boldsymbol{b}$. In the case of the minimum cost flow problem, we can set the value of one variable in the dual problem freely. Then we define the dual polyhedron of the minimum cost flow problem $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ as $\{\boldsymbol{y} \in \mathbb{R}^d | \boldsymbol{y}A \leq \boldsymbol{c} \text{ and } y_d = 0\}$.

In this case, $N_{\boldsymbol{c}}$ has a special property as follows.

Lemma 3.1 *Let* $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_s\}$ *be the minimal vectors such that* $N_{\boldsymbol{c}} = \bigcup_{i=1}^{s} (\boldsymbol{p}_i + \mathbb{N}^n)$*. Then all* $\boldsymbol{p}_1, \dots, \boldsymbol{p}_s$ *are 0-1 vectors for any cost vector* $\boldsymbol{c} \in \mathbb{R}^n_{\geq 0}$*.*

Proof: In [8], we showed that reduced Gröbner basis for I_A with respect to any $\boldsymbol{c}$ corresponds to the set of circuits in N. This implies that the exponent vectors of reduced Gröbner basis are 0-1 vectors. □

When $N_{\boldsymbol{c}}$ is generated by 0-1 vectors, the set $S(O_{\boldsymbol{c}})$ are obtained by all of the maximal faces of $\Delta_{\boldsymbol{c}}$.

Lemma 3.2 ([7]) *Let* $\{\boldsymbol{p}_1, \dots, \boldsymbol{p}_s\}$ *be the minimal vectors such that* $N_{\boldsymbol{c}} = \bigcup_{i=1}^{s} (\boldsymbol{p}_i + \mathbb{N}^n)$*. Then* $\boldsymbol{p}_1, \dots, \boldsymbol{p}_s$ *are all 0-1 vectors if and only if* $S(O_{\boldsymbol{c}}) = \{((0, \dots, 0), \sigma) \mid \sigma \text{ is a maximal face of } \Delta_{\boldsymbol{c}}\}$*.*

The edges in the optimum flow of uncapacitated minimum cost flow problem define a forest [1]. Therefore, with the fact the dimension of *cone*(A) equals $d - 1$, the next proposition is implied by Lemma 2.3, 3.1 and 3.2.

Proposition 3.3 *Any standard pair of* $O_{\boldsymbol{c}}$ *is* $((0, \dots, 0), \sigma)$*, where* $\sigma \subseteq \{1, \dots, n\}$ *and* σ *is a spanning tree of the network* N*.*

Since there is one-to-one correspondence between the standard pairs $((0, \dots, 0), *)$ of $O_{\boldsymbol{c}}$ and the vertices of dual polyhedron of the minimum cost flow problem, Algorithm 2.3 for the minimum cost flow problem $IP_{A,\boldsymbol{c}}(\boldsymbol{b})$ is a variant of the enumeration of dual feasible bases.

Example 2.2 (continued.) *Figure 3 shows the dual polyhedron* $P_{\boldsymbol{c}}$ *of the minimum cost flow problems for* $\boldsymbol{c}$ *with* $c_1 + c_3 > c_2$*.*

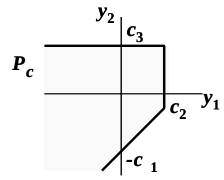

Figure 3: The dual polyhedron $P_{\boldsymbol{c}}$ of the minimum cost flow problems for the network N in Example 2.2

4 Arithmetic degree of acyclic directed graphs

In the case that the network is the acyclic tournament graph, the minimum and maximum arithmetic degrees can be obtained using some results about Gröbner bases and hypergeometric functions. When the network is an acyclic directed graph, these results give upper and lower bounds for the arithmetic degree. Since each standard pair of the minimum cost flow problem corresponds to the vertex of the dual polyhedron for the linear relaxation of the minimum cost flow problem, these results also give upper and lower bounds for the number of vertices of the dual polyhedron. In this section we show that, for the acyclic tournament graph with d vertices, the minimum arithmetic degree is 1 and the maximum arithmetic degree is $\frac{1}{d}\binom{2(d-1)}{d-1}$, which is the $(d-1)$-Catalan number.

We assume that the vertices of acyclic directed graph have labels $1, \ldots, d$ such that each edge (i, j) is oriented from the vertex i to the vertex j. We denote c_{ij} the cost of the edge (i, j) and x_{ij} the variable corresponds to the edge (i, j). The number of edges equals $n = \binom{d}{2}$.

4.1 Minimum arithmetic degree of acyclic tournament graphs

We remark again that the genericity of $\boldsymbol{c}$ implies that, the arithmetic degree of $O_{\boldsymbol{c}}$ is equal to or greater than 1. We show that there exists a cost vector $\boldsymbol{c}$ for which the arithmetic degree equals 1 for any d.

Theorem 4.1 *For the acyclic tournament graph with d vertices, the minimum arithmetic degree of $O_{\boldsymbol{c}}$ which $\boldsymbol{c}$ varies in all generic cost vectors equals* 1.

To show this theorem, we use a characterization of one $N_{\boldsymbol{c}}$.

Lemma 4.2 *When $\boldsymbol{c} = (c_{ij})_{1\le i<j\le d}$ satisfies that $c_{ij} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$ for any $i < j-1$, then $N_{\boldsymbol{c}} = \bigcup_{i<j-1}(\boldsymbol{e}_{ij} + \mathbb{N}^n) \subset \mathbb{N}^n$.*

Proof: In [8], we showed that reduced Gröbner basis $G_{\boldsymbol{c}}$ for I_A with respect to the above $\boldsymbol{c}$ is $\{x_{ij} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} \mid 1 \le i < j-1 \le d-1\}$. Especially, the set of initial terms of $G_{\boldsymbol{c}}$ equals $\{x_{ij} \mid 1 \le i < j-1 \le d-1\}$, which implies that $N_{\boldsymbol{c}} = \bigcup_{i<j-1}(\boldsymbol{e}_{ij} + \mathbb{N}^n)$. □

Proof of Theorem 4.1: Because of Lemma 4.2, $\boldsymbol{a} = (a_{ij})_{1\le i<j\le d} \in \mathbb{N}^n$ is in $O_{\boldsymbol{c}}$ for $\boldsymbol{c}$ as in Lemma 4.2 if and only if $a_{ij} = 0$ for any (i, j) such that $j - i > 1$. The set of all such points coincides $((0, \ldots, 0), \{(1, 2), (2, 3), \ldots, (d-1, d)\})$. Thus only this pair is a standard pair of $O_{\boldsymbol{c}}$. □

4.2 Maximum arithmetic degree of acyclic tournament graphs

To show the maximum arithmetic degree, we use another integer programming problem defined from the original integer programming problem. For a general matrix $A \in \mathbb{Z}^{d\times n}$, we define another matrix $A' \in \mathbb{Z}^{(d+1)\times(n+1)}$ as

$$\begin{aligned} A' &:= \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & & & & 0 \\ & & A & & \vdots \\ & & & & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ & & & & 0 \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n & \vdots \\ & & & & 0 \end{pmatrix}. \end{aligned} \tag{2}$$

Let $\boldsymbol{a}'_i = \binom{1}{\boldsymbol{a}_i}$ for $1 \le i \le n$ and $\boldsymbol{a}'_{n+1}$ be the $n+1$-th column vector of A'. We remark that $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_n, \boldsymbol{a}'_{n+1}$ span an affine hyperplane.

We define another family $IP_{A',(\boldsymbol{c},0)}$ of integer programming problems

$$IP_{A',(\boldsymbol{c},0)}(\boldsymbol{b},\beta) := \textit{minimize}\left\{ \boldsymbol{c}\cdot\boldsymbol{x} \;\middle|\; \begin{array}{l} A'\binom{\boldsymbol{x}}{x_{n+1}} = \binom{\beta}{\boldsymbol{b}}, \\ \binom{\boldsymbol{x}}{x_{n+1}} \in \mathbb{N}^{n+1} \end{array} \right\}$$

as $\binom{\beta}{\boldsymbol{b}}$ varies in $\{A'\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{N}^{n+1}\}$. Similarly as $O_{\boldsymbol{c}}$, let $O'_{(\boldsymbol{c},0)} \subset \mathbb{N}^{n+1}$ be the set of all optimal solutions to all programs in $IP_{A',(\boldsymbol{c},0)}$, and $N'_{(\boldsymbol{c},0)} := \mathbb{N}^{n+1} \setminus O'_{(\boldsymbol{c},0)}$.

The next proposition is obtained by the result for general monomial ideals[12], but here we directly show using the properties of toric ideals.

Proposition 4.3 $(\boldsymbol{a},\sigma)$ $(\boldsymbol{a} \in \mathbb{N}^n,\ \sigma \subseteq \{1,\ldots,n\})$ *is a standard pair of* $O_{\boldsymbol{c}}$ *if and only if* $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right)$ *is a standard pair of* $O'_{(\boldsymbol{c},0)}$.

Proof: We first show that $(\boldsymbol{a},\sigma) \subseteq O_{\boldsymbol{c}}$ if and only if $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right) \subseteq O'_{(\boldsymbol{c},0)}$. Suppose that $(\boldsymbol{a},\sigma) \subseteq O_{\boldsymbol{c}}$ and choose any $\binom{\boldsymbol{u}}{k} \in \left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right)$. If there exist any other $\boldsymbol{v} \in \mathbb{N}^n$ and non-negative integer l such that $A'\binom{\boldsymbol{v}}{l} = A'\binom{\boldsymbol{u}}{k}$ and $\binom{\boldsymbol{v}}{l} \neq \binom{\boldsymbol{u}}{k}$, then $A\boldsymbol{u} = A\boldsymbol{v}$, and $\boldsymbol{c}\cdot\boldsymbol{u} < \boldsymbol{c}\cdot\boldsymbol{v}$ since $\boldsymbol{u} \in O_{\boldsymbol{c}}$. Therefore, $\binom{\boldsymbol{u}}{k}$ is the optimal solution to $IP_{A',(\boldsymbol{c},0)}(A\boldsymbol{u},\beta)$ with $\beta = \sum_{i=1}^n u_i + k$. If there does not exist such $\binom{\boldsymbol{v}}{l}$, then clearly $\binom{\boldsymbol{u}}{k}$ is the optimal for this integer programming problem. This shows that $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right) \subseteq O'_{(\boldsymbol{c},0)}$.

Conversely, suppose that $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right) \subseteq O'_{(\boldsymbol{c},0)}$ and choose any $\boldsymbol{u} \in (\boldsymbol{a},\sigma)$. If there exists some $\boldsymbol{v} \in \mathbb{N}^n$ such that $A\boldsymbol{v} = A\boldsymbol{u}$, then $A'\binom{\boldsymbol{u}}{p} = A'\binom{\boldsymbol{v}}{q}$ for any non-negative integers p, q such that $p-q = \sum_{i=1}^n v_i - \sum_{i=1}^n u_i$. Then $\boldsymbol{c}\cdot\boldsymbol{u} < \boldsymbol{c}\cdot\boldsymbol{v}$ because $\binom{\boldsymbol{u}}{p} \in \left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right) \subseteq O'_{(\boldsymbol{c},0)}$ and $(\boldsymbol{c},0)\cdot\binom{\boldsymbol{u}}{p} < (\boldsymbol{c},0)\cdot\binom{\boldsymbol{v}}{q}$. Therefore, $\boldsymbol{u}$ is the optimal solution to $IP_{A,\boldsymbol{c}}(A\boldsymbol{u})$. If there does not exist such $\boldsymbol{v}$, then clearly $\boldsymbol{u}$ is the optimal for this integer programming problem. Thus $(\boldsymbol{a},\sigma) \subseteq O_{\boldsymbol{c}}$.

If $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right)$ is a standard pair of $O'_{(\boldsymbol{c},0)}$, then $\left(\binom{\boldsymbol{a}}{0}, \sigma\cup\{n+1\}\right) \not\subseteq \left(\binom{\boldsymbol{a}'}{0}, \tau\cup\{n+1\}\right)$ for any other $\left(\binom{\boldsymbol{a}'}{0}, \tau\cup\{n+1\}\right)$ which satisfies the conditions (i) and (ii) in Definition 2.1 for

$O'_{(\boldsymbol{c},0)}$. Since if $(\boldsymbol{a},\sigma) \subset (\boldsymbol{a}',\tau)$ for any other $(\boldsymbol{a}',\tau)$ which satisfies (i) and (ii) in Definition 2.1 for $O_{\boldsymbol{c}}$, then $\left(\binom{\boldsymbol{a}}{0}, \sigma \cup \{n+1\}\right)$ must be contained in $\left(\binom{\boldsymbol{a}'}{0}, \tau \cup \{n+1\}\right)$, which contradicts the assumption. Thus $(\boldsymbol{a},\sigma)$ is a standard pair of $O_{\boldsymbol{c}}$.

On the other hand, if $\left(\binom{\boldsymbol{a}}{0}, \sigma \cup \{n+1\}\right)$ is not a standard pair of $O'_{(\boldsymbol{c},0)}$, then there exists some $\left(\binom{\boldsymbol{a}'}{k}, \tau'\right)$ such that $\left(\binom{\boldsymbol{a}'}{k}, \tau'\right)$ contains $\left(\binom{\boldsymbol{a}}{0}, \sigma \cup \{n+1\}\right)$ and $\left(\binom{\boldsymbol{a}'}{k}, \tau'\right)$ satisfies (i) and (ii) in Definition 2.1 for $O'_{(\boldsymbol{c},0)}$. Then $n+1 \in \tau'$, and therefore $k=0$ by (i) in Definition 2.1. Therefore, $(\boldsymbol{a}',\tau)$ where $\tau := \tau' \backslash \{n+1\}$ contains $(\boldsymbol{a},\sigma)$ and satisfies (i) and (ii) in Definition 2.1. Thus $(\boldsymbol{a},\sigma)$ is not a standard pair of $O_{\boldsymbol{c}}$. This completes the proof. $\square$

Example 2.2 (continued.) *For this A, enlarged matrix A' is*

$$A' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix}.$$

For $\boldsymbol{c}$ with $c_1+c_3 > c_2$, the standard pairs of $O'_{(\boldsymbol{c},0)}$ are $((0,0,0,0),\{1,2,4\})$ and $((0,0,0,0),\{2,3,4\})$. On the other hand, for $\boldsymbol{c}$ with $c_1+c_3 < c_2$, the standard pairs of $O'_{(\boldsymbol{c},0)}$ are $((0,0,0,0),\{1,2,3\})$ and $((0,0,0,0),\{1,3,4\})$. In this case, the only standard pair $((0,0,0,0),\{1,3,4\})$ satisfies the condition in Proposition 4.3, which corresponds to the standard pair $((0,0,0),\{1,3\})$ of $O_{\boldsymbol{c}}$.

Theorem 4.4 *For the acyclic tournament graph with d vertices, the maximum arithmetic degree of $O_{\boldsymbol{c}}$ which $\boldsymbol{c}$ varies in all generic cost vectors equals*

$$C_{d-1} := \frac{1}{d}\binom{2(d-1)}{d-1},$$

which is the $(d-1)$-th Catalan number.

To show this theorem, we prepare some lemmas. The first is by Gelfand et al. [5] which studies about some hypergeometric function.

Lemma 4.5 ([5]) *Let A' be the enlarged matrix (2) for the incidence matrix A of the acyclic tournament graph with d vertices, and $conv(A')$ be the convex hull of $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$. Then the normalized volume of $conv(A')$ equals the $(d-1)$-th Catalan number C_{d-1}.*

Given a regular triangulation $\Delta'_{(\boldsymbol{c},k)}$ of $conv(A')$, the normalized volume of $conv(A')$ equals the sum of normalized volume for all maximal faces in $\Delta'_{(\boldsymbol{c},k)}$. Since the column vectors in $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$ span an affine hyperplane, the normalized volume of $conv(A')$ gives the number of standard pairs of $O'_{(\boldsymbol{c},k)}$ which correspond to the maximal faces of $\Delta'_{(\boldsymbol{c},k)}$ by Lemma 2.3 (iii).

Lemma 4.6 *When $\boldsymbol{c} = (c_{ij})_{1 \le i < j \le d}$ satisfies that*

- $c_{ij} + c_{jk} > c_{ik}$ *for any $i < j < k$ and*
- $c_{ik} + c_{jl} > c_{il} + c_{jk}$ *for any $i < j < k < l$,*

then

$$N_{\boldsymbol{c}} = \bigcup_{1\le i<j<k\le d} ((\boldsymbol{e}_{ij}+\boldsymbol{e}_{jk})+\mathbb{N}^n) \cup \bigcup_{1\le i<j<k<l\le d} ((\boldsymbol{e}_{ik}+\boldsymbol{e}_{jl})+\mathbb{N}^n) \subset \mathbb{N}^n.$$

Proof: In [8], we showed that reduced Gröbner basis $G_{\boldsymbol{c}}$ for I_A with respect to the above $\boldsymbol{c}$ is $\{x_{ij}x_{jk} - x_{ik} \mid 1 \le i < j < k \le d\} \cup \{x_{ik}x_{jl} - x_{il}x_{jk} \mid 1 \le i < j < k < l \le d\}$. Especially, the set of initial terms of $G_{\boldsymbol{c}}$ equals $\{x_{ij}x_{jk} \mid 1 \le i < j < k \le d\} \cup \{x_{ik}x_{jl} \mid 1 \le i < j < k < l \le d\}$, which implies that $N_{\boldsymbol{c}} = \bigcup_{1\le i<j<k\le d}((\boldsymbol{e}_{ij}+\boldsymbol{e}_{jk})+\mathbb{N}^n) \cup \bigcup_{1\le i<j<k<l\le d}((\boldsymbol{e}_{ik}+\boldsymbol{e}_{jl})+\mathbb{N}^n)$. □

Proof of Theorem 4.4: Because of Proposition 3.3, any standard pair of $O_{\boldsymbol{c}}$ is $((0,\dots,0),\sigma)$ for any $\boldsymbol{c}$, and corresponds to the standard pair $((0,\dots,0,0),\sigma\cup\{n+1\})$ of $O'_{(\boldsymbol{c},0)}$. Especially, $\sigma\cup\{n+1\}$ is a maximal face of $\Delta'_{(\boldsymbol{c},0)}$. Therefore,

$$\begin{aligned} \textit{arith-deg}\,(O_{\boldsymbol{c}}) &= |\{((0,\dots,0),\sigma) \in S\,(O_{\boldsymbol{c}})\}| \\ &= \left|\left\{((0,\dots,0),\sigma\cup\{n+1\}) \in S\left(O'_{(\boldsymbol{c},0)}\right)\right\}\right| \\ &\le \sum_{\tau} \left|\left\{(*,\tau) \in S\left(O'_{(\boldsymbol{c},0)}\right)\right\}\right| \qquad (3) \\ &= \text{normalized volume of } conv(A') \\ &= C_{d-1} \end{aligned}$$

where the sum in (3) is taken over all maximal faces τ of $\Delta_{(\boldsymbol{c},0)}$.

Because of Proposition 3.3 and Lemma 4.6, for $\boldsymbol{c}$ as in Lemma 4.6 $((0,\dots,0),\sigma)$ is a standard pair of $O_{\boldsymbol{c}}$ if and only if σ is a spanning tree of the acyclic tournament graph which satisfies the following two conditions:

(A) there are no $1 \le i < j < k \le d$ such that both (i,j) and (j,k) are edges in σ, and

(B) there are no $1 \le i < j < k < l \le d$ such that both (i,k) and (j,l) are edges in σ.

The number of spanning trees of the complete graph with d vertices which satisfies **(A)** and **(B)** are known to be equal to the $(d-1)$-th Catalan number (e.g. see [9]). □

We remark that the Catalan number equals $\frac{4^n}{\sqrt{\pi}n^{3/2}}\left(1+O\left(\frac{1}{n}\right)\right)$ (e.g. see [3]).

Example 2.2 (continued.) $\boldsymbol{c} = (3,1,2)$ *gives an example achieving the maximum arithmetic degree since* $C_2 = \frac{1}{3}\binom{4}{2} = 2$. *On the other hand,* $\boldsymbol{c} = (1,4,2)$ *gives an example achieving the minimum arithmetic degree.*

4.3 Arithmetic degree of acyclic directed graphs

Since any acyclic directed graph G with d vertices can be seen as the subgraph of acyclic tournament graph $\widetilde{G}$ with same number of vertices by deleting some edges, the incidence matrix A of G is a submatrix of that $\widetilde{A}$ of $\widetilde{G}$ by deleting the columns which correspond to the deleted edges. Let A' (resp. $\widetilde{A}'$) be the enlarged matrix (2) of A (resp. $\widetilde{A}$). Then the normalized volume of $conv(A')$ is equal to or less than that of $conv(\widetilde{A})$, which implies that the arithmetic degree for $IP_{A,\boldsymbol{c}}$ for any cost vector $\boldsymbol{c}$ is equal to or less than the maximum arithmetic degree for the acyclic tournament graph, which is the $(d-1)$-th Catalan number.

5 Conclusion

In this paper, we showed the bounds for the number of vertices of the dual polyhedra for the minimum cost flow problems on the acyclic directed graphs, using the results from Gröbner bases and hypergeometric functions. We also showed two examples for the acyclic tournament graphs, one achieves the minimum arithmetic degree 1, and the other achieves the maximum arithmetic degree C_{d-1}.

On the other hand, the arithmetic degrees for cyclic directed graphs are not known. Similar approach may be able to attack to this analysis, which should be a future work.

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