On the structure of weak interlaced bilattice $\mathcal{K}$(L)

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Abstract

We study fundamental properties of weak interlaced bilattices $\mathcal{K}(L)$ and show that for any weak interlaced bilattice $\mathcal{W}$ there exists a lattice L such that $\mathcal{W}$ can be embedded into a weak interlaced bilattice $\mathcal{K}(L)$. Hence, any interlaced bilattice can be embedded into the weak interlaced bilattice $\mathcal{K}(L)$ for some lattice L.

1 Introduction

It is well-known the Kleene's 3-valued logic in the field of multiple-valued logics. The logic has three values *false, true*, and $\perp$ (*unknown*) as truth values. These values have two informal orderings concerning "amount of knowledge" and "degree of truth". For example, if we think of a certain proposition such as *Goldbach's conjecture* assigned $\perp$ as truth value, then it is possible that we can conclude the truth value of the proposition as *true* or *false* with increasing knowledge. Thus in the ordering of knowledge, $\perp$ is smaller than *true* and *false*. A sentence with $\perp$ is between *false* and *true* in the ordering of degree of truth. In this way it can be considered that the three valued logic has two orderings. Belnap ([**?**]), Ginsberg([**?**]), and others proposed concept of a *bilattice* which has two orderings and proved some fundamental results ([**?**, **?**, **?**]). It is shown by Fitting ([**?**]) that bilattices can give a uniform semantics for many lanuages of logic programming. Since then the theory of bilattices is a hot reserach field.

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On the other hand, as in *Fuzzy logics*, a truth value can be taken as a closed interval $[a, b]$. Let L be a lattice and $\mathcal{K}(L)$ be the set of all closed intervals of L. In this case we also define two orderings. For $[a, b], [c, d] \in \mathcal{K}(L)$, if $[a, b] \subseteq [c, d]$ then the knowledge in $[a, b]$ is greater than that in $[c, d]$. Thus we set $[a, b] \sqsubseteq_k [c, d]$ if $[a, b] \subseteq [c, d]$. Likewise we also define $[a, b] \sqsubseteq_t [c, d]$ if $a \leq c$ and $b \leq d$, because $[c, d]$ is greater than $[a, b]$ in the ordering degree of truth. The structure $\mathcal{K}(L) =< \mathcal{K}(L), \sqsubseteq_t, \sqsubseteq_k >$ which precise definition is given below has the property of *weak interlaced bilattice*.

In [**?**, **?**], Fitting, Font and Moussavi have investigated the strucutre of $\mathcal{K}(L)$ and proved that if L is a bounded lattice, then $\mathcal{K}(L)$ is a weak interlaced bilattice ([**?**]). Now does the converse hold?, that is, is there a lattice L such that $\mathcal{W} \cong \mathcal{K}(L)$ for every weak interlaced bilattice $\mathcal{W}$?

Clearly we answer "No". Because we have a simple counterexample. Let $\mathcal{B}$ be a weak interlaced bilattice with 5 elements, for example, a set $\{0, p, \perp, q, 1\}$ with $0 \leq_t p \leq_t \perp \leq_t q \leq_t 1$, $\perp \leq_k p \leq_k 0$ and $\perp \leq_k q \leq_k 1$. It is obvious that $\mathcal{B}$ is a weak interlaced bilattice. Suppose that there is a lattice L such that $\mathcal{B} \cong \mathcal{K}(L)$. If $|L| \geq 3$, then there exists an element $a \in L$ such that $0 < a < 1$. For that element we have $[0, 0], [0, a], [0, 1], [a, 1], [a, a], [1, 1] \in \mathcal{K}(L)$ and $|\mathcal{K}(L)| \geq 6$. Since $|\mathcal{B}| = 5$, it must be $|L| \leq 2$. But, in this case, we have $|\mathcal{K}(L)| \leq 3$. This means that there is no lattice L such that $\mathcal{B} \cong \mathcal{K}(L)$.

Now we settle a more general question.

> **Question** : For every weak interlaced bilattice $\mathcal{W}$, is there a lattice L such that $\mathcal{W}$ can be embedded to $\mathcal{K}(L)$?

In this note we study properties of $\mathcal{K}(L)$ and answer the question.

2 Definition of $\mathcal{K}(L)$

We define a structure $\mathcal{K}(L)$ for any lattice L. Let $L = (L, \leq)$ be a lattice and $K(L)$ be the set of all closed intervals of L, that is,

$$K(L) = \{[a, b] | a \leq b, a, b \in L\}$$
$$[a, b] = \{x | a \leq x \leq b\}.$$

For any $[a, b], [c, d] \in K(L)$, we define two orderings $\sqsubseteq_t, \sqsubseteq_k$ on $K(L)$ as follows :

$$[a,b] \sqsubseteq_t [c,d] \Longleftrightarrow a \leq c, b \leq d$$
$$[a,b] \sqsubseteq_k [c,d] \Longleftrightarrow a \leq c, b \geq d$$

We set $\mathcal{K}(L) =< K(L), \sqsubseteq_t, \sqsubseteq_k>$. It is obvious from definition that $[0,0]$ ($[1,1]$) is the minimum (maximum) element with respect to $\sqsubseteq_t$. On the other hand, while $[0,1]$ is the minimum element, there is no maximum element with respect to the ordering $\sqsubseteq_k$. This means that $\mathcal{K}(L)$ is a lattice with respect to $\sqsubseteq_t$ and is a semi-lattice concering $\sqsubseteq_k$. Four operators $\sqcap_t, \sqcup_t, \sqcap_k, \sqcup_k$ are defined by

$$\begin{aligned}
\inf_{\sqsubseteq_t}\{a,b\} &= a \sqcap_t b \\
\sup_{\sqsubseteq_t}\{a,b\} &= a \sqcup_t b \\
\inf_{\sqsubseteq_k}\{a,b\} &= a \sqcap_k b \\
\sup_{\sqsubseteq_k}\{a,b\} &= a \sqcap_k b \quad \text{(if it is defined)}
\end{aligned}$$

Next we give definitions of an interlaced bilattice and of a weak interlaced bilattice. A relational system $< B, \leq_t, \leq_k>$ is called an *interlaced bilattice* if it satisfies

1. B is a non-empty set
2. $< B, \leq_t>$, $< B, \leq_k>$ are bounded lattices and satisfy
   (a) $x \leq_t y \Longrightarrow x \otimes z \leq_t y \otimes z,\ x \oplus z \leq_t y \oplus z$
   (b) $x \leq_k y \Longrightarrow x \wedge z \leq_k y \wedge z,\ x \vee z \leq_k y \vee z$

where four operators are defined by

$$\begin{aligned}
\inf_{\leq_t}\{x,y\} &= x \wedge y \\
\sup_{\leq_t}\{x,y\} &= x \vee y \\
\inf_{\leq_k}\{x,y\} &= x \otimes y \\
\sup_{\leq_k}\{x,y\} &= x \oplus y
\end{aligned}$$

By $0(1)$, we mean the minimum (maximum) element with respect to the ordering $\leq_t$. We also denote by $\perp(\top)$ the minimum (maximum) element concering to $\leq_k$.

A map $\neg$ from B into itself is called a *negation* if

$x \leq_t y \Longrightarrow \neg y \leq_t \neg x$

$x \leq_k y \Longrightarrow \neg x \leq_k \neg y$

$\neg\neg x = x.$

For lattices $L_1 =< L_1, \wedge_1, \vee_1 >$ and $L_2 =< L_2, \wedge_2, \vee_2 >$, we define operations $\wedge, \vee, \otimes, \oplus$ on the product $L_1 \times L_2$: For $(a, b), (c, d) \in L_1 \times L_2$,

$$\begin{aligned}(a, b) \wedge (c, d) &= (a \wedge_1 c, b \vee_2 d)\\ (a, b) \vee (c, d) &= (a \vee_1 c, b \wedge_2 d)\\ (a, b) \otimes (c, d) &= (a \wedge_1 c, b \wedge_2 d)\\ (a, b) \oplus (c, d) &= (a \vee_1 c, b \vee_2 d).\end{aligned}$$

The structure $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is called a *Ginsberg product*. There are some fundamental results about the structure :

Proposition 1 (Fitting). *If L_1, L_2 are bounded lattices then the Ginsberg product $L_1 \odot L_2 =< L_1 \times L_2, \wedge, \vee, \otimes, \oplus >$ is an interlaced bilattice. Espectially, $L \odot L$ is an interlaced bilattice with negation $\neg$, where $\neg$ is defined by $\neg(a, b) = (b, a)$.*

It is proved that the converse holds by Avron ([**?**]).

Proposition 2. *[Avron] For any interlaced bilattice $\mathcal{B}$, there are bounded lattices L_1, L_2 such that $\mathcal{B} \cong L_1 \odot L_2$. In particular, for any interlaced bilattice $\mathcal{B}$ with negation, there is a bounded lattice L such that $\mathcal{B} \cong L \odot L$.*

It is clear from definition that orderings $\sqsubseteq_t, \sqsubseteq_k$ on $\mathcal{K}(L)$ are the same as $\leq_t, \leq_k$ on Ginsberg product $L \odot L$, respectively :

$\sqsubseteq_t$ in $\mathcal{K}(L) \Longleftrightarrow \leq_t$ in $L \odot L$

$\sqsubseteq_k$ in $\mathcal{K}(L) \Longleftrightarrow \leq_k$ in $L \odot L$

Hence in the following we use the same symbols $\wedge, \vee, \otimes, \oplus$ in $\mathcal{K}(L)$ and in $L \odot L$.

Next we give a definition of a *weak interlaced bilattice* according to Font ([**?**]). A structure $\mathcal{W} =< W, \leq_t, \leq_k >$ is called a *weak interlaced bilattice* if

1. $< W, \leq_t >$Flattice

2. $< W, \leq_k >$Fmeet semilattice

3. $a \leq_k b, c \leq_k d \Longrightarrow a \wedge c \leq_k b \wedge d, a \vee c \leq_k b \vee d$

4. $a \leq_t b, c \leq_t d \Longrightarrow a \otimes c \leq_t b \otimes d$,

5. $a \leq_t b, c \leq_t d \Longrightarrow a \oplus c \leq_t b \oplus d$ if $a \oplus c$ and $b \oplus d$ exist.

3 Properties of weak interlaced bilattices

For any weak interlaced bilattice $\mathcal{W}$, if we define

$$L_1 = \{x \in \mathcal{W} \mid x \leq_k 0\} = [\perp, 0]_k$$
$$L_2 = \{x \in \mathcal{W} \mid x \leq_k 1\} = [\perp, 1]_k,$$

then we have

Proposition 3.

$$L_1 = [\perp, 0]_k = [0, \perp]_t$$
$$L_2 = [\perp, 1]_k = [\perp, 1]_t$$

Proof. Let $x \in [\perp, 0]_k$. Since $\perp \leq_k x \leq_k 0$, we have $\perp \vee \perp \leq_k x \vee \perp \leq_k 0 \vee \perp$ by definition of weak interlaced bilattice. From $\perp \vee \perp = 0 \vee \perp = \perp$, it follows that $x \vee \perp = \perp$ and hence that $x \leq_t \perp$. This means $[\perp, 0]_k \subseteq [0, \perp]_t$.

Conversely, suppose $x \in [0, \perp]_t$. If we put $u = 0 \otimes x$, then it is clear that $u \leq_k 0$ and $u \leq_k x$. Since $0 \leq_t x$, we have $0 \otimes x \leq_t x \otimes x = x$ and hence $u \leq_t x$. It follows from $\perp \leq_k u$ that $x \wedge \perp \leq_k x \wedge u$. Since $x \leq_t \perp$, we also have $x \wedge \perp = x$. On the other hand, since $u \leq_t x$, we get $u \wedge x = u$. Theses imply that $x \leq_k u$ and hence that $x = u$. Thus we have $x \leq_k 0$. Namely, we have $[0, \perp]_t \subseteq [\perp, 0]_k$.

The second equation can be proved similarly.

□

The result implies that L_1 and L_2 are lattices with ordering $\leq_1$ and $\leq_2$ in $\mathcal{B}$, respectively, where $\leq_1$ and $\leq_2$ are defined by

$$\leq_1 = \leq_t = \geq_k$$
$$\leq_2 = \leq_t = \leq_k$$

Thus we can consider the Ginsberg product $L_1 \odot L_2$, which becomes an *interlaced bilattice*. Moreover we can prove

Proposition 4. *Let $\mathcal{W}$ be any weak interlaced bilattice. For any $x \in \mathcal{W}$, we have*
$x = (x \otimes 0) \oplus (x \otimes 1) = (x \wedge \bot) \vee (x \vee \bot)$

Proof. See Avron [**?**] Cor.3.8 □

Now we investigate a realtion between a weak interlaced bilattice $\mathcal{W}$ and an interlaced bilattice $L_1 \odot L_2$ constructed by $\mathcal{W}$.

Lemma 1. *A map $\xi : \mathcal{W} \to L_1 \times L_2$ defined by $\xi(x) = (x \otimes 1, x \otimes 0) = (x \vee \bot, x \wedge \bot)$ is an embedding.*

This means that

Theorem 1. *Any weak interlaced bilattice can be embedded into an interlaced bilattice.*

4 Answer to the question

In this section we give a positive answer to the question above. Since any weak interlaced bilattice $\mathcal{W}$ can be embedded to an interlaced bilattice, it sufficies to show that any interlaced bilattice of a form $L_1 \odot L_2$ is embeddable into a weak interlaced bilattice $\mathcal{K}(L)$ for some lattice L. Because, from proposition **??**, every interlaced bilattice has a form of $L_1 \odot L_2$ for some lattices L_1, L_2. Let $L_1 \odot L_2$ be any interlaced bilattice and L be a set $(L_1 \times \{0\}) \cup (L_2 \times \{1\})$. We define an order $\sqsubseteq$ on L. For any element $(a, i), (b, j) \in L$, we define

$$(a, i) \sqsubseteq (b, j) \iff i < j \text{ or } i = j \text{ and } a \leq b$$

It is easy to show that the relation $\sqsubseteq$ is a partially order on L and that

$$(a,i) \wedge (b,j) = \inf\{(a,i),(b,j)\} = \begin{cases} (a \wedge b, i) & \text{if } i = j \\ (a,i) & \text{if } i < j \\ (b,j) & \text{if } i > j \end{cases}$$

$$(a,i) \vee (b,j) = \sup\{(a,i),(b,j)\} = \begin{cases} (a \vee b, i) & \text{if } i = j \\ (b,j) & \text{if } i < j \\ (a,i) & \text{if } i > j \end{cases}$$

Hence L is a lattice with this order. Let $\mathcal{K}(L)$ be the set of all elements $[(a,i),(b,j)]$ such that $(a,i) \sqsubseteq (b,j)$ for $(a,i),(b,j) \in L$. In this case, four operators $\wedge, \vee, \otimes, \oplus$ on $\mathcal{K}(L)$ are defined as follows:

$$
\begin{aligned}
[(a,i),(b,j)] \wedge [(a',i'),(b',j')] &= [(a,i) \wedge (a',i'), (b,j) \wedge (b',j')] \\
[(a,i),(b,j)] \vee [(a',i'),(b',j')] &= [(a,i) \vee (a',i'), (b,j) \vee (b',j')] \\
[(a,i),(b,j)] \otimes [(a',i'),(b',j')] &= [(a,i) \wedge (a',i'), (b,j) \vee (b',j')] \\
[(a,i),(b,j)] \oplus [(a',i'),(b',j')] &= [(a,i) \vee (a',i'), (b,j) \wedge (b',j')]
\end{aligned}
$$

Of course, the last equation is defined when $(a,i) \vee (a',i') \leq (b,j) \wedge (b',j')$.

Now we define a map $\xi : L_1 \odot L_2 \rightarrow \mathcal{K}(L)$ by

$$\xi(a,b) = [(a,0),(b,1)]$$

It is obvious that ξ is well-defined and injective. We only show that ξ is a homomorphism. We only think of two cases. For the case of $(a,b) \wedge (a',b')$, we have

$$
\begin{aligned}
\xi((a,b) \wedge (a',b')) &= \xi(a \wedge a', b \vee b') \\
&= [(a \wedge a', 0), (b \vee b', 1)] \\
&= [(a,0) \wedge (a',0), (b,1) \vee (b',1)] \\
&= [(a,0),(b,1)] \otimes [(a',0),(b',1)] \\
&= \xi(a,b) \otimes \xi(a',b')
\end{aligned}
$$

For another case of $(a,b) \oplus (a',b')$, we also have

$$
\begin{aligned}
\xi((a,b) \oplus (a',b')) &= \xi(a \vee a', b \vee b') \\
&= [(a \vee a', 0), (b \vee b', 1)] \\
&= [(a,0) \vee (a',0), (b,1) \vee (b',1)] \\
&= [(a,0),(b,1)] \vee [(a',0),(b',1)] \\
&= \xi(a,b) \vee \xi(a',b')
\end{aligned}
$$

Hence the map $\xi : L_1 \odot L_2 \rightarrow \mathcal{K}(L)$ is an embedding, that is,

Theorem 2. *For every interlaced bilattice $L_1 \odot L_2$, there exists a lattice L such that it is embedded into a weak interlaced bilattice $\mathcal{K}(L)$.*

From these results, we have have a main theorem.

Theorem 3. *Every interlaced bilattice $\mathcal{W}$ can be embedded into a weak interlaced bilattice $\mathcal{K}(L)$ for some lattice L.*

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