Covering radius problems for non self-dual binary codes

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1 Introduction

Determination of the covering radius and complete coset distribution of a code are important problems in the coding theory. When the code is linear and self-dual there have been major progresses [1],[3],[21],[47],[48],[49],[50]. When the code is linear but not self-dual there is much room for further progress. In this report we first explain how self-dual case is treated, and next we try to find an approach to reach satisfactory results in non self-dual case.

2 Standard Definitions from Binary Codes and the Statements of the problem

Let $\mathbf{F}_2 = GF(2)$ be the field of 2 elements. Let $V = \mathbf{F}_2^n$ be the vector space of dimension n over $\mathbf{F}_2$. A linear $[n,k]$ code $\mathbf{C}$ is a vector subspace of V of dimension k. An element $\mathbf{x}$ in $\mathbf{C}$ is called a codeword of $\mathbf{C}$. In V , the inner product , which is denoted by $(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y}$ in V, is defined as usual. Two codes $\mathbf{C}_1$ and $\mathbf{C}_2$ are said to be equivalent if and only if after a suitable change of coordinate positins of $\mathbf{C}_1$ all the codewords in both codes coincide. The dual code $\mathbf{C}^\perp$ of $\mathbf{C}$ is defined by

$$\mathbf{C}^\perp = \{\mathbf{u} \in V \mid (\mathbf{u},\mathbf{v}) = \mathbf{0} \quad \forall \mathbf{v} \in \mathbf{C}\}.$$

The code $\mathbf{C}$ is called self-orthogonal if it satisfies $\mathbf{C} \subseteq \mathbf{C}^\perp$, and the code $\mathbf{C}$ is called self-dual if it satisfies $\mathbf{C} = \mathbf{C}^\perp$.

Self-dual codes exist only if $n \equiv 0 \pmod 2$ and $k = \dfrac{n}{2}$

Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

be a vector in V , then the Hamming weight $wt(\mathbf{x})$ of the vector $\mathbf{x}$ is defined to be the number of i's such that $x_i \neq 0$. The Hamming distance d on V is also defined by $d(\mathbf{x},\mathbf{y}) = wt(\mathbf{x}-\mathbf{y})$.

Let $\mathbf{C}$ be a code ,then the minimum distance d of the code $\mathbf{C}$ is defined by

$$\begin{aligned} d &= \mathrm{Min}_{\mathbf{x},\mathbf{y}\in\mathbf{C},\mathbf{x}\neq\mathbf{y}} d(\mathbf{x},\mathbf{y}) \\ &= \mathrm{Min}_{\mathbf{x}\in\mathbf{C},\mathbf{x}\neq\mathbf{0}} wt(\mathbf{x}). \end{aligned}$$

Again let $\mathbf{x}$ be a vector in V, then the Hamming sphere $S_t(\mathbf{x})$ of radius t with center $\mathbf{x}$ is defined by

$$S_t(\mathbf{x}) = \{\mathbf{y} \in V \mid d(\mathbf{y},\mathbf{x}) \leq t\}.$$

The covering radius $t(\mathbf{C})$ of the code $\mathbf{C}$ is defined to be the smallest integer t such that the equation

$$V = \bigcup_{\mathbf{x}\in\mathbf{C}} S_t(\mathbf{x})$$

holds. The covering radius of a code $\mathbf{C}$ is also defined by another way.
A vector $\mathbf{v}$ in a coset U in $V/\mathbf{C}$ is a coset leader if $\mathbf{v}$ satisfies

$$wt(\mathbf{v}) \leq wt(\mathbf{z}) \text{ for all } \mathbf{z} \in U.$$

Then the covering radius of $\mathbf{C}$ is the weight of a coset leader of greatest weight.

A reformulation of this problem is given by

Proposition 1 *It holds that*

$$t(\mathbf{C}) = \max_{\mathbf{u}\in\mathbf{F}_2^n}(\min_{\mathbf{z}\in\mathbf{u}+\mathbf{C}} wt(\mathbf{z})).$$

A more precise problem is that for a given code $\mathbf{C}$ determine all the coset weight enumerators (or equally the coset weight distributions) $W_U(X)$.

Let $\mathbf{C}$ be a self-dual binary $[n, \frac{n}{2}]$ code, then the weight $wt(\mathbf{x})$ of each codeword $\mathbf{x}$ in $\mathbf{C}$ is an even number. Further, if the weight of each codeword $\mathbf{x}$ in $\mathbf{C}$ is divisible by 4, then the code is called a doubly even binary code. It is known that doubly even self-dual binary codes $\mathbf{C}$ exist only when the length n of $\mathbf{C}$ is a multiple of 8.

Let $\mathbf{u} = (u_1, u_2, \cdots, u_n), \mathbf{v} = (v_1, v_2, \cdots, v_n)$ be any pair of vectors in $\mathbf{F}_2^n$, then the number of common 1's of the corresponding coordinates for $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} * \mathbf{v}$. This is called the intersection number of $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{u} * \mathbf{u}$ is nothing else $wt(\mathbf{u})$.

Let $\mathbf{C}$ be a doubly even self-dual binary $[n, \frac{n}{2}]$ code. The inhomogeneous weight enumerator $W_{\mathbf{C}}(X)$ of the code $\mathbf{C}$ is defined by

$$W_{\mathbf{C}}(X) = \sum_{\mathbf{v}\in\mathbf{C}} X^{wt(\mathbf{v})}.$$

If we denote by a_r the number of the codewords $\mathbf{v}$ of weight r in $\mathbf{C}$, then $W_{\mathbf{C}}(X)$ can be rewritten as

$$W_{\mathbf{C}}(X) = \sum_{r=0}^{n} a_r X^r.$$

For example, the extended Hamming code e_8 given in the table has the weight enumerator

$$W_{e_8}(X) = 1 + 14X^4 + X^8.$$

We also give $W_{\mathcal{G}_{24}}(X)$ explicitly :

$$W_{\mathcal{G}_{24}}(X) = 1 + 759X^8 + 2576X^{12} + 759X^{16} + X^{24},$$

where $\mathcal{G}_{24}$ is the extended binary Golay code of length 24.
Usually, the weight enumerator is expressed in the homogeneous form :

$$W_{\mathbf{C}}(x, y) = \sum_{r=0}^{n} a_r x^{n-r} y^r.$$

Again, the homogeneous form of $W_{e_8}(X)$ is

$$W_{e_8}(x, y) = x^8 + 14x^4y^4 + y^8$$

But these two polynomials are mutually transformable into another forms by the rules

$$y^n W_{\mathbf{C}}(\frac{x}{y}) = W_{\mathbf{C}}(x, y),$$

and

$$y^{-n} W_{\mathbf{C}}(x, y) = W_{\mathbf{C}}(\xi), \text{ with } \xi = \frac{x}{y},$$

where n is the length of the code $\mathbf{C}$.
A celebrated MacWilliams identity says :

Theorem 1 *Let $W_{\mathbf{C}}(X)$ be the weight enumerator of a binary code, then the following identity holds :*

$$W_{\mathbf{C}^\perp}(X) = \frac{1}{\mid \mathbf{C} \mid}(1 + X)^n W_{\mathbf{C}}(\frac{1 - X}{1 + X}).$$

The homogeneous version of the above theorem is

Theorem 2

$$W_{\mathbf{C}^\perp}(x, y) = \frac{1}{\mid \mathbf{C} \mid} W_{\mathbf{C}}(x + y, x - y) \tag{1}$$

The coset weight enumerator of a coset U in a code $\mathbf{C}$ is defined to be a polynomial in X given by

$$W_U(X) = \sum_{\mathbf{z} \in U} X^{wt(\mathbf{z})}.$$

We give an example.

Let e_8 be the Hamming code, and $u_0 = 0$ the zero vector. For the coset $U = u_0 + e_8$ we obtain

$$W_U(X) = 1 + 14X^4 + X^8$$

For the coset $U = u_1 + e_8$ with $wt(u_1) = 1$ we obtain

$$W_U(X) = X + 7X^3 + 7X^5 + X^7$$

3 Definition of Jacobi polynomials for Binary codes.

Throughout this section we let $\mathbf{C}$ be a self-dual code of length n. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{F}_2^n$, then $\mathbf{u} * \mathbf{v}$ is the intersection number of $\mathbf{u}$ and $\mathbf{v}$.

Definition 1 Jacobi polynomials for codes

Jacobi polynomial $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ for $\mathbf{C}$ with respect to $\mathbf{v} \in \mathbf{F}_2^n$ is defined by

$$Jac(\mathbf{C}, \mathbf{v} \mid X, Z) = \sum_{\mathbf{u} \in \mathbf{C}} X^{\mathbf{u}*\mathbf{u}} Z^{\mathbf{u}*\mathbf{v}}.$$

The homogeneous Jacobi polynomial is defined by

$$Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) = \sum_{\mathbf{u} \in \mathbf{C}} x^{n-wt(\mathbf{v})-wt(\mathbf{u})+\mathbf{u}*\mathbf{v}} y^{wt(\mathbf{u})-\mathbf{u}*\mathbf{v}} u^{wt(\mathbf{v})-\mathbf{u}*\mathbf{v}} v^{\mathbf{u}*\mathbf{v}},$$

, where x, y, u, v are algebraically independent variables over $\mathbb{C}$. The connection between inhomogeneous Jacobi and homogeneous Jacobi is described by

$$Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) = x^{n-wt(\mathbf{v})} u^{wt(\mathbf{v})} Jac_i(\mathbf{C}, \mathbf{v}, yx, xvyu),$$

and

$$Jac_i(\mathbf{C}, \mathbf{v}, X, Z) = Jac(\mathbf{C}, \mathbf{v}, 1, X, 1, XZ).$$

4 Some Fundamental Theorems

Let $\mathbb{C}[X]$ be the ring of all polynomials in one determinate X with coefficients in $\mathbb{C}$, the ring of rational integers. We denote by $\mathcal{W}[X]$ the subring of $\mathbb{C}[X]$ generated by the inhomogeneous weight enumerators attached to doubly even self-dual binary codes. The first proposition is a restatement of a well-known result :

Theorem 3 (Gleason) *$\mathcal{W}[X]$ has two polynomial generators $W_{e_8}(X)$ (the weight enumerazor of the Hamming code e_8) and $W_{\mathcal{G}_{24}}(X)$ (the weight enumerator of the Golay [24,12,8] code $\mathcal{G}_{24}$.*

Let $\mathbb{C}[X, Z]$ be the ring of all polynomials in two determinates X and Z, with coefficients in $\mathbb{C}$. Let $\mathbb{J}[X, Z]$ be the subring of $\mathbb{C}[X, Z]$ generated by Jacobi polynomials attached to self-dual binary codes.

Let $\mathbf{C}$ be a self-dual $[n, \frac{n}{2}]$ code containing all 1 vector. Let $\mathbf{v}$ be any vector in $\mathbf{F}^n$.

One of our basic theorems is :

Theorem 4 *Let $\mathbf{C}$ be a binary code of length n and $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ a Jacobi polynomial for the code $\mathbf{C}$ with any binary vector $\mathbf{v} \in \mathbf{F}_2^n$, then it holds that*

$$\begin{aligned} & Jac(\mathbf{C}^{\perp}, \mathbf{v} \mid X, Z) \\ = \ & \frac{1}{|\mathbf{C}|}(1+X)^n \left(\frac{1+XZ}{1+X}\right)^{wt(\mathbf{v})} \\ & \times Jac(\mathbf{C}, v \mid \frac{1-X}{1+X}, \frac{(1-XZ)(1+X)}{(1+XZ)(1-X)}). \end{aligned}$$

There is a homogeneous version of the above theorem :

Theorem 5 *Let* $\mathbf{C}$ *be a self-dual code of length* n*, then* $Jac(\mathbf{C},\mathbf{v},x,y,u,v)$ *satisfies*

$$Jac(\mathbf{C},\mathbf{v},\frac{x+y}{\sqrt{2}},\frac{x-y}{\sqrt{2}},\frac{u+v}{\sqrt{2}},\frac{u-v}{\sqrt{2}}) = Jac(\mathbf{C},\mathbf{v},x,y,u,v), \tag{2}$$

5 Algebraic Study of Jacobi Polynomials for Doubly Even Self-Dual Codes

Let $\mathbf{C}$ be a doubly even self-dual code of length n, then the weight enumerator of $\mathbf{C}$ satisfies

$$W_{\mathbf{C}}(x,iy) = W_{\mathbf{C}}(x,y). \tag{3}$$

We consider the following linear actions of the variables x and y :

$$\sigma_1 : \begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

and

$$\sigma_2 : \begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

We let G_d denote the group of linear transformations generated by σ_1 and σ_2. G_1 is a finite unitary group ofgenerated by reflections and No. 9 in the list of Shephard-Todd [58], and its order is 192. The properties (1), (3) of $W_{\mathbf{C}}(x,y)$ imply that $W_{\mathbf{C}}(x,y)$ belongs to the ring of invariance for the group G_1, $\mathbb{C}[x,y]^{G_1}$.
The homogeneous Jacobi polynomial has the invariance properties (2) and

(4). $$Jac(\mathbf{C},\mathbf{v},x,y,u,v) = Jac(\mathbf{C},\mathbf{v},x,iy,u,iv)$$

We put

$$\tilde{\sigma_1} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & & \\ 1 & -1 & & O \\ & O & 1 & 1 \\ & & 1 & -1 \end{pmatrix},$$

$$\tilde{\sigma_2} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & & \\ 0 & i & & O \\ & O & 1 & 0 \\ & & 0 & i \end{pmatrix}$$

and $\hat{G}_d =< \tilde{\sigma_1}, \tilde{\sigma_2} >$. Then we see that

$$Jac(\mathbf{C},\mathbf{v},x,y,u,v) \in \mathbb{C}[x,y,u,v]^{\hat{G}_d},$$

where $\mathbb{C}$ is the field of complex numbers.
Molien series for $\hat{G}_d$ and for a subgroup of index 2 is computed in [**?**]. We determined the ring $\mathbb{C}[x,y,u,v]^{\hat{G}_d}$ in [4].

6 Algebraic Study of Jacobi Polynomials for Singly Even Self-Dual Codes

If we put

$$\tilde{\sigma_1} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} & O \\ O & \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \end{pmatrix},$$

$$\tilde{\sigma_3} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} & O \\ O & \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix}$$

and $\hat{G}_s =< \tilde{\sigma_1}, \tilde{\sigma_3} >$. Then we see that

$$Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) \in \mathbb{C}[x, y, u, v]^{\hat{G}_s},$$

where $\mathbb{C}$ is the field of complex numbers.
Molien series for $\hat{G}_s$ ($|\hat{G}_s| = 16$) is given by

$$\begin{aligned} \Phi_{\hat{G}_s}(t) &= \frac{1}{\hat{G}_s} \sum_{g \in \hat{G}_s} \frac{1}{\det(I_4 - tg)} \\ &= \sum_{k \geq 0} \dim((\mathbb{C}[x, y, u, v]^{\hat{G}_s})_k) t^k, \\ &= \frac{(t^{16} + t^{14} + t^{12} + t^{10} + 8t^8 + t^6 + t^4 + t^2 + 1)}{(t^{20} - 2t^{18} + t^{16} - 2t^{12} + 4t^{10} - 2t^8 + t^4 - 2t^2 + 1)} \\ &= \frac{1 + t^2 + t^4 + t^6 + 8t^8 + t^{10} + t^{12} + t^{14} + t^{16}}{(1 - t^2)^2 (1 - t^8)^2}. \end{aligned}$$

Here $(\mathbb{C}[x, y, u, v]^{\hat{G}})_k$ denotes the vector space consisting of the k-th homogeneous part of the ring $\mathbb{C}[x, y, u, v]^{\hat{G}}$. The Molien series $\hat{G}$ is of great help to us in forming the explicit basis of the space $(\mathbb{C}[x, y, u, v]^{\hat{G}})_k$, sometimes indispensable.

In [49] we proved

Proposition 2 *The primary invariants (i.e. the denominators) for $\mathbb{C}[x, y, u, v]^{\hat{G}}$ can be replaced by*

$$\begin{aligned} \varphi_{2,0} &= x^2 + y^2 \\ \varphi_{2,2} &= u^2 + v^2 \\ \varphi_{8,0} &= x^8 + 14x^4y^4 + y^8 \\ \varphi_{8,8} &= u^8 + 14u^4v^4 + v^8, \end{aligned}$$

and the secondary invariants (the numerators) for $\mathbb{C}[x,y,u,v]^{\hat{G}}$ *can be replaced by*

$$\begin{array}{lcl} & & 1 \\ \varphi_{2,1}^{j} & = & (xu+yv)^{j}, 1 \le j \le 8 \\ \varphi_{8,1} & = & x^7u + 7x^4y^3v + 7x^3y^4u + vy^7, \\ \varphi_{8,2} & = & x^6u^2 + 3x^4y^2v^2 + 8x^3y^3uv + 3x^2y^4u^2 + y^6v^2, \\ \varphi_{8,3} & = & u^3x^5 + v^3yx^4 + 6v^2uy^2x^3 + 6vu^2y^3x^2 + u^3y^4x + v^3y^5 \\ \varphi_{8,4} & = & u^4x^4 + 4v^3uyx^3 + 6v^2u^2y^2x^2 + 4vu^3y^3x + v^4y^4 \\ \varphi_{8,5} & = & x^3u^5 + x^3v^4u + 6v^3u^2yx^2 + 6v^2u^3y^2x + y^3vu^4 + y^3v^5 \\ \varphi_{8,6} & = & x^2u^6 + 3x^2v^4u^2 + 8v^3u^3yx + 3y^2v^2u^4 + y^2v^6 \\ \varphi_{8,7} & = & xu^7 + 7xu^3v^4 + 7yu^4v^3 + yv^7, \end{array}$$

As a consequence of the above proposition we have

Theorem 6 *Let the notations be as above, then we have*

$$\begin{aligned} & \mathbb{C}[x,y,u,v]^{\hat{G}} \\ = \;& \mathcal{R} \bigoplus \varphi_{2,1}\mathcal{R} \bigoplus \varphi_{2,1}^{2}\mathcal{R} \bigoplus \varphi_{2,1}^{3}\mathcal{R} \bigoplus \varphi_{2,1}^{4}\mathcal{R} \\ & \bigoplus \varphi_{2,1}^{5}\mathcal{R} \bigoplus \varphi_{2,1}^{6}\mathcal{R} \bigoplus \varphi_{2,1}^{7}\mathcal{R} \bigoplus \varphi_{2,1}^{8}\mathcal{R} \\ & \bigoplus \varphi_{8,1}\mathcal{R} \bigoplus \varphi_{8,2}\mathcal{R} \bigoplus \varphi_{8,3}\mathcal{R} \bigoplus \varphi_{8,4}\mathcal{R} \\ & \bigoplus \varphi_{8,5}\mathcal{R} \bigoplus \varphi_{8,6}\mathcal{R} \bigoplus \varphi_{8,7}\mathcal{R}, \end{aligned}$$

where $\mathcal{R} = \mathbb{C}[\ \varphi_{2,0},\ \varphi_{2,2},\ \varphi_{8,0},\ \varphi_{8,8}\ ]$.

7 A Description of the Process of Determining the Complete Coset Weight Distributions of Codes

Now we explain how the complete coset weight distributions of a code $\mathbf{C}$ can be obtained. Once we could know $Jac_i(\mathbf{C},\mathbf{v},X,Z)$, we can know $W_{\mathbf{C}+\mathbf{v}}(X)$ by the following (c.f. [47])

Proposition 3 *One has*

$$W_{\mathbf{v}+\mathbf{C}}(X) = X^{wt(\mathbf{v})} Jac_i(\mathbf{C},\mathbf{v}|\ X, X^{-2}).$$

The multiplicities of the coset weight enumerators are controled by the following proposition (c.f. also [47])

Proposition 4 *Let* $\mathbf{C}$ *be a binary code of length* n*. We are under the assumption :*
(∗) in a binary linear code $\mathbf{C}$ *any two different rigid Jacobi polynomials derive (via the process in Proposition 8) different coset weight enumerators .*

Let $\mathbf{R}_k$ *be the set of rigid vectors of a fixed weight* k *with equal rigid Jacobi polynomials* $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$. *Let*

$$\begin{aligned} & X^k Jac_i(\mathbf{C}, \mathbf{v} \mid X, X^{-2}) \\ = & \ W_{\mathbf{v}+\mathbf{C}}(X) \\ = & \ a_k X^k + a_{k+1} X^{k+1} + \cdots \end{aligned}$$

be its common coset weight enumerator, then the number of the different cosets $\mathbf{S}$ *in* $\mathbf{F}_2^n/\mathbf{C}$ *such that*

$$W_{\mathbf{S}}(X) = W_{\mathbf{v}+\mathbf{C}}(X)$$

is given by $\dfrac{\mid \mathbf{R}_k \mid}{a_k}$.

8 Relation between covering radius of codes and Jacobi polynomials

Our next result is

Theorem 7 *Le* $\mathbf{C}$ *be a self-dual binary code of length* n. *We take any vector* $\mathbf{v}$ *in* $\mathbf{F}_2^n$. *If* U *is the coset in* $\mathbf{F}_2^n/\mathbf{C}$ *to which* $\mathbf{v}$ *belongs, then we have*

$$W_U(X) = X^{wt(\mathbf{v})} Jac(C, \mathbf{v} \mid X, X^{-2}),$$

where $Jac(C, \mathbf{v} \mid X, Z)$ *is the Jacobi polynomial for the code* $\mathbf{C}$ *with respect to* $\mathbf{v}$.

We now give a precise information on the relation between the multiplicity of Jacobi polynomials and the multiplicity of the coset weight enumerators. To do this we need a definition.

Definition 3 : Let $\mathbf{C}$ be a binary code of length n. Let $\mathbf{v}$ be a vector in $\mathbb{F}_2^n$. The vector $\mathbf{v}$ is called rigid (with respect to $\mathbf{C}$) if $wt(\mathbf{v})$ is minimal in its coset $\mathbf{v} + \mathbf{C}$ i.e. if $\mathbf{v}$ satisfies

$$wt(\mathbf{v}) = \min_{\mathbf{x} \in \mathbf{v}+\mathbf{C}} wt(\mathbf{x}).$$

A vector $\mathbf{v}$ in $\mathbb{F}_2^n$ is fragile (with rewspect to $\mathbf{C}$) if $\mathbf{v}$ is not rigid. A Jacobi polynomial $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$ is rigid if $\mathbf{v}$ is rigid and ,fragile if $\mathbf{v}$ is fragile.

Theorem 8 *Let* $\mathbf{C}$ *be a binary code of length* n. *Let* $\mathbf{R}_k$ *be the set of rigid vectors of a fixed weight* k *with equal rigid Jacobi polynomials* $Jac(\mathbf{C}, \mathbf{v} \mid X, Z)$. *Let*

$$\begin{aligned} X^k Jac(\mathbf{C}, \mathbf{v} \mid X, X^{-2}) &= W_{\mathbf{v}+\mathbf{C}}(X) \\ &= aX^k + bX^{k+1} + \cdots \end{aligned}$$

be its common coset weight enumerator, then the number of the cosets $\mathbf{S}$ *in* $\mathbb{F}_2^n/\mathbf{C}$ *such that*

$$W_{\mathbf{S}}(X) = W_{\mathbf{v}+\mathbf{C}}(X)$$

holds, is given by $\dfrac{\mathbf{R}_k}{a}$.

9 A summary of the covering radius problem for binary linear self-dual codes

A table of covering radius of self-dual binary codes
(extremal or optimal)

n	d_I	d_{II}	$t(C_I)$	$t(C_{II})$	*# of eq. cl. of* C_I	*# of eq. cl. of* C_{II}
2	2	/	1	/	1	/
4	2	/	2	/	1	/
6	2	/	3	/	1	/
8	2	4	4	2	1	1
10	2	/	5	/	2	/
12	4	/	3	/	1	/
14	4	/	3	/	1	/
16	4	4	4	4	1	2
18	4	/	$5, 4$	/	2	/
20	4	/	$4, 5, 6$	/	7	/
22	6	/	7	/	1	/
24	6	8	5	4	1	1
26	6	/	*not yet*	/	1	/
28	6	/	*not yet*	/	3	/
30	6	/	*not yet*	/	13	/
32	8	8	6	6	3	5
34	6	/	*not yet*	/	≥ 200	/
36	8	/	7	/	≥ 2	/
38	8	/	*not yet*	/	≥ 3	/
40	8	8	$7, 8$	$7, 8$	≥ 22	≥ 1000

In the above table
/ implies the non-existence case, d_I (resp. d_{II}) is the maximal value of the minimal distance of the type I (resp. type II) code of given length n . Here we quote some values from [49], [55].

A table of that who determined the complete coset weight distributions of doubly even self-dual binary extremal codes

length	*who*	*year*	*comment*
8	*folklore*		
16	*Assmus – Pless* [1]	1983	
24	*folklore?*	*once upon a time*	
32	*Assmus – Pless* [1]	1983	*only* XQR_{32}
32	*Camion and al.*[10]	1993	*five codes*
40	*Ozeki* [47]	2000	
48	*Delsarte* [21]	1973	*contains an error*
56	*Ozeki* [48]	2000	
64		*not yet*	*possible*
72		*not yet*	*existence is not known*
80		*not yet*	*possible*
88		*not yet*	*possible*

10 Non self-dual binary linear code case

In the determination of the covering radius and the complete coset weight distribution of binary self-dual codes there have been explored a variety of methods. Compared to self-dual binary linear codes the corresponding problems in the class of non self-dual binary linear codes $\mathbf{C}$ are less explored. So far the main trend in this area is to give upper bounds and lower bounds for $t(\mathbf{C})$ [6], [13], [15]. From the point of practical applications there are two important classes of non self-dual linear binary codes: Reed-Muller codes and BCH codes. The study of the covering radius and the complete coset weight distribution for these classes of codes would be important. Therefore we pose a question concerning these problems. One may remark that the use of electronic computers in these problems is inevitable but the computer is of secondary importance, and the primary importance lies in the mathmatical insights that break the difficulties of the mechanism.
Question 1. What can we do the covering radius and the complete coset weight distribution for these classes of codes along with the theory of invariants. Can we extend our method using invariant theory of finite groups to these cases?

We give one possible approach to this question.
Experience in the study shows that the followings are useful tools:
(i) MacWilliams identity for the weight enumerators and the Jacobi polynomials,
(ii) Delsarte's relation [21] page 414, formula (3.5),
(iii) polarization process in the invariant theory,
(iv) Jacobi polynomials for binary codes,
(v) the invariant theory for finite groups.

Let $\mathcal{R}_s$ be the ring for the group of invariance for the self-dual biary codes. We consider the $\mathcal{R}_s$-module generated by the Jacobi polynomials for non self-dual binary codes.
Let $\mathbf{C}$ be a binary $[n, k]$ code, then $\mathbf{C}^\perp$ is a binary $[n, n-k]$ code. We assume that $k \leq \frac{n}{2}$.
The following picture may be helpful.

$$\begin{array}{ccc}
\mathbf{C} & & \mathbf{C}^\perp \\
W_{\mathbf{C}}(x,y) & & W_{\mathbf{C}^\perp}(x,y) \\
J_{n,m} & \rightarrow & J_{n,n-m} \\
\Cup & & \Cup \\
Jac(\mathbf{C}, \mathbf{v}, x, y, u, v) & \mapsto & Jac(\mathbf{C}^\perp, \mathbf{v}, x, y, u, v) \\
J_{n,m} & \cong & J_{n,n-m} \\
\dim J_{n,m} & = & \dim J_{n,n-m}
\end{array}$$

Problem 1. Determine the dimension
Problem 2. Describe or reduce the complexity of computing in $J_{n,m}$ by means of $J_{n',m'}$ with $n' \leq n, m' \leq m$.
Problem 3. Describe a necessary and sufficient condition that a polynomial in $J_{n,m}$ to be rigid in terms of the conditions in the polynomials in $J_{n,n-m}$.
Localized settings of the above problems.
E.g. $J_{2^h,m}(R(r,h))$: the linear space generated by Jacobi polynomials for $R(r,h)$.

At present we have not fully explored the problems described above along our line of ideas. The examples in the following sections will be a symptom for the success, or they do not show any real difficuties that we do not see now.

11 Elements of Reed-Muller codes

$R(r, m)$: r-th order Reed-Muller code of Length 2^m

We now give briefly the definition of Reed-Muller codes.

We denote by (1^n)(resp. (0^n)) the all one(resp. zero) vector of length n. We use the symbol **ab** to denote the concatenation of two vectors **a** and **b**, and the symbol $\mathbf{a}^\ell$ to denote the concatenation of ℓ **a**'s. For example, we see that

$$(1^3) = (111), \quad (1^2 0^2) = (1100)\text{and}(1010)^2 = (10101010)$$

and so on. We put

$$\begin{aligned}
\mathbf{x}_0 &= (1^{2^m}), \\
\mathbf{x}_1 &= (0^{2^{m-1}} 1^{2^{m-1}}), \\
\mathbf{x}_2 &= ((0^{2^{m-2}} 1^{2^{m-2}})^2 \\
&\vdots \\
\mathbf{x}_m &= ((01))^{2^{m-1}}).
\end{aligned}$$

If **x** and **y** are two vectors of the same length, then we denote by **xy** another vector of identical length obtained from the component-wise product of those two vectors. We call it the vector product of **x** and **y**. Likewise we can define the vector product of arbitrary number of vectors of the same length.

The r-th order Reed-Muller code $R(r, m)$ of length 2^m is defined to be a binary linear code spanned by vectors

$$\prod_{1 \le i_1 \le i_2 \le \cdots \le i_h \le m} \mathbf{x}_{i_1} \mathbf{x}_{i_2} \cdots \mathbf{x}_{i_h}, \ h \le r$$

and by $\mathbf{x}_0$.

We give an example. Let $m = 4$, then we see that

$$\mathbf{x}_0 = (1111111111111111),$$

$$\begin{aligned}
\mathbf{x}_1 &= (0000000011111111), \\
\mathbf{x}_2 &= (0000111100001111), \\
\mathbf{x}_3 &= (0011001100110011),
\end{aligned}$$

and

$$\mathbf{x}_4 = (0101010101010101).$$

Further we see that

$$\begin{aligned}
\mathbf{x}_1\mathbf{x}_2 &= (0000000000001111), \\
\mathbf{x}_1\mathbf{x}_3 &= (0000000000110011), \\
\mathbf{x}_1\mathbf{x}_4 &= (0000000001010101), \\
\mathbf{x}_2\mathbf{x}_3 &= (0000001100000011), \\
\mathbf{x}_2\mathbf{x}_4 &= (0000010100000101),
\end{aligned}$$

and

$$\mathbf{x}_3\mathbf{x}_4 = (0001000100010001).$$

Thus the generator matrix of the first order Reed-Muller code of length $2^4 = 16$ $R(1,4)$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and the generator matrix of the second order Reed-Muller code of length $2^4 = 16$ $R(2,4)$ is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\mathcal{C}_1 = R(1,5)$

Weight enumerator of $\mathcal{C}$

$$1 + 62X^{16} + X^{32}$$

Jacobi polynomials of $\mathcal{C}$

When $wt(\mathbf{v}) = 1$ the multiplicity $= 32$.

$$\begin{aligned} &Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad 1 + X^{16}(31Z + 31) + X^{32}Z \end{aligned}$$

$$\begin{aligned} &XJac(\mathcal{C}_1, \mathbf{v} \mid X, X^{-2}) = \\ &\quad X + 31X^{15} + 31X^{17} + X^{31} \end{aligned}$$

(2)When $wt(\mathbf{v}) = 2$ the multiplicity $= 496$.

$$\begin{aligned} &Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad 1 + X^{16}(15Z^2 + 32Z + 15) + X^{32}Z^2 \end{aligned}$$

and

(3)When $wt(\mathbf{v}) = 3$ the multiplicity $= 4960$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(7Z^3 + 24Z^2 + 24Z + 7) + X^{32}Z^3$$

(4)-(1) When $wt(\mathbf{v}) = 4$ the multiplicity $= 34720$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(3Z^4 16Z^3 + 24Z^2 + 16Z + 3) + X^{32}Z^4$$

(4)-(2)When $wt(\mathbf{v}) = 4$ the multiplicity $= 1240$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(7Z^4 + 48Z^2 + 7) + X^{32}Z^4$$

Rigid Jacobi polynomials

(5)-(1)When $wt(\mathbf{v}) = 5$ the multiplicity $= 34720$

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(3Z^5 + 4Z^4 + 24Z^3 + 24Z^2 + 4Z + 3) + +X^{32}Z^5$$

(5)-(2)When $wt(\mathbf{v}) = 5$ the multiplicity $= 166656$

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(Z^5 + 10Z^4 + 20Z^3 + 20Z^2 + 10Z + 1) + X^{32}Z^5$$

(6)-(1)When $wt(\mathbf{v}) = 6$, the multiplicity $= 416640$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(Z^6 + 4Z^5 + 14Z^4 + 24Z^3 + 14Z^2 + 4Z + 1) + X^{32}Z^6$$

(6)-(2)When $wt(\mathbf{v}) = 6$, the multiplicity $= 17360$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(3Z^6 + 12Z^4 + 32Z^3 + 12Z^2 + 3) + X^{32}Z^6$$

(6)-(3)When $wt(\mathbf{v}) = 6$, the multiplicity $= 27776$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ 1 + X^{16}(Z^6 + 30Z^4 + 30Z^2 + 1) + X^{32}Z^6$$

(6)-(4)When $wt(\mathbf{v}) = 6$, the multiplicity $= 444416$.

$$Jac(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(6Z^5 + 15Z^4 + 20Z^3 + 15Z^2 + 6Z) + X^{32}Z^6$$

(7)-(1)When $wt(\mathbf{v}) = 7$, the multiplicity $= 4960$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(3Z^7 + 28Z^4 + 28Z^3 + 3) + X^{32}Z^7 +$$

(7)-(2)When $wt(\mathbf{v}) = 7$, the multiplicity $= 416640$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(Z^7 + 2Z^6 + 6Z^5 + 22Z^4 + 22Z^3 6Z^2 + 2Z + 1) + X^{32}Z^7 +$$

(7)-(3)When $wt(\mathbf{v}) = 7$, the multiplicity $= 277760$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(Z^7 + 12Z^5 + 18Z^4 + 18Z^3 12Z^2 + 1) + X^{32}Z^7 +$$

(7)-(4)When $wt(\mathbf{v}) = 7$, the multiplicity $= 2222080$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(3Z^6 + 9Z5 + 19Z^4 + 19Z^3 + 9Z^2 + 3Z) + X^{32}Z^7 +$$

(7)-(5)When $wt(\mathbf{v}) = 7$, the multiplicity $= 444416$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$1 + X^{16}(Z^6 + 15Z^5 + 15Z^4 + 15Z^3 + 15Z^2 + 1) + X^{32}Z^7 +$$

(8)-(1)When $wt(\mathbf{v}) = 8$, the multiplicity $= 119040$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$+X^{16}(Z^8 + 2Z^7 + 14Z^5 + 28Z^4 + 14Z^3 + 2Z + 1) + X^{32}Z^8$$

(8)-(2)When $wt(\mathbf{v}) = 8$, the multiplicity $= 624960$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$+X^{16}(Z^8 + 4Z^6 + 16Z^5 + 20Z^4 + 16Z^3 + 4Z^2 + 1) + X^{32}Z^8$$

(8)-(3)When $wt(\mathbf{v}) = 8$, the multiplicity $= 3333120$

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) =$$
$$+X^{16}(2Z^7 + 4Z^6 + 14Z^5 + 22Z^4 + 14Z^3 + 4Z2 + 2Z) + X^{32}Z^8$$

(8)-(4)When $wt(\mathbf{v}) = 8$, the multiplicity = 1666560

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(8Z^6 + 16Z^5 + 14Z^4 + 16Z^3 + 8Z^2) + X^{32}Z^8$$

(8)-(5)When $wt(\mathbf{v}) = 8$, the multiplicity = 4444160

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(Z^7 + 6Z^6 + 15Z^5 + 18Z^4 + 15Z^3 + 6Z^2 + Z) + X^{32}Z^8$$

(8)-(6)When $wt(\mathbf{v}) = 8$, the multiplicity = 52080

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(Z^8 + 8Z^6 + 44Z^4 + 8Z^2 + 1) + X^{32}Z^8$$

(8)-(7)When $wt(\mathbf{v}) = 8$, the multiplicity = 277760

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(12Z^6 + 38Z^4 + 12Z^2) + X^{32}Z^8$$

(8)-(8)When $wt(\mathbf{v}) = 8$, the multiplicity = 620

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(3Z^8 + 56Z^4 + 3) + X^{32}Z^8$$

Rigid Jacobi polynomials

(9)-(1)When $wt(\mathbf{v}) = 9$, the multiplicity = 952320

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(2Z^8 + Z^7 + 7Z^6 + 21Z^5 + 21Z^4 + 7Z^3 + Z^2 + 2Z) + X^{32}Z^9$$

(9)-(2)When $wt(\mathbf{v}) = 9$, the multiplicity = 8888320

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(3Z^7 + 13Z^6 + 15Z^5 + 15Z^4 + 13Z^3 + 3Z^2) + X^{32}Z^9$$

(9)-(3)When $wt(\mathbf{v}) = 9$, the multiplicity = 833280

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(Z^8 + 4Z^7 + 4Z^6 + 22Z^5 + 22Z^4 + 4Z^3 + 4Z^2 + Z) + X^{32}Z^9$$

(9)-(4)When $wt(\mathbf{v}) = 9$, the multiplicity = 9999360

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(Z^8 + 2Z^7 + 10Z^6 + 18Z^5 + 18Z^4 + 10Z^3 + 2Z^2 + Z) + X^{32}Z^9$$

(9)-(5)When $wt(\mathbf{v}) = 9$, the multiplicity = 6666240

$$Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ +X^{16}(5Z^7 + 7Z^6 + 19Z^5 + 19Z^4 + 7Z^3 + 5Z^2) + X^{32}Z^9$$

(10)-(1)When $wt(\mathbf{v}) = 10$, the multiplicity $= 83328$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(5Z^8 + 10Z^6 + 32Z^5 + 10Z^4 + 5Z^2) + X^{32}Z^{10}\end{aligned}$$

(10)-(2)When $wt(\mathbf{v}) = 10$, the multiplicity $= 20832000$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(Z^8 + 8Z^7 + 14Z^6 + 16Z^5 + 14Z^4 + 8Z^3 + Z^2) + X^{32}Z^{10}\end{aligned}$$

(10)-(3)When $wt(\mathbf{v}) = 10$, the multiplicity $= 7221760$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(3Z^8 + 4Z^7 + 12Z^6 + 24Z^5 + 12Z^4 + 4Z^3 + 3Z^2) + X^{32}Z^{10}\end{aligned}$$

(10)-(4)When $wt(\mathbf{v}) = 10$, the multiplicity $= 19998720$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(2Z^8 + 6Z^7 + 13Z^6 + 20Z^5 + 13Z^4 + 6Z^3 + 2Z^2) + X^{32}Z^{10}\end{aligned}$$

(10)-(5)When $wt(\mathbf{v}) = 10$, the multiplicity $= 2666496$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(10Z^7 + 15Z^6 + 12Z^5 + 15Z^4 + 10Z^3) + X^{32}Z^{10}\end{aligned}$$

(10)-(6)When $wt(\mathbf{v}) = 10$, the multiplicity $= 1984000$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(3Z^8 + 28Z^6 + 28Z^4 + 3Z^2) + X^{32}Z^{10}\end{aligned}$$

(11)-(1)When $wt(\mathbf{v}) = 11$, the multiplicity $= 42163968$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(5Z^8 + 10Z^7 + 16Z^6 + 16Z^5 + 10Z^4 + 5Z^3) + X^{32}Z^{11}\end{aligned}$$

(11)-(2)When $wt(\mathbf{v}) = 11$, the multiplicity $= 6666240$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(3Z^8 + 16Z^7 + 12Z^6 + 12Z^5 + 16Z^4 + 3Z^3) + X^{32}Z^{11}\end{aligned}$$

(12)-(1)When $wt(\mathbf{v}) = 12$, the multiplicity $= 2499840$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(11Z^8 + 16Z^7 + 8Z^6 + 16Z^5 + 11Z^4) + X^{32}Z^{12}\end{aligned}$$

(12)-(2)When $wt(\mathbf{v}) = 12$, the multiplicity $= 3513664$

$$\begin{aligned}&Jac^{(1)}(\mathcal{C}_1, \mathbf{v} \mid X, Z) = \\ &\quad +X^{16}(15Z^8 + 32Z^6 + 15Z^4) + X^{32}Z^{12}\end{aligned}$$

We summarize the above computation as a table below. The table is a reconstruction of Berlekamp-Welch analysis [5].

coset wt	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	mul
coset wt	32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	
0	1																62	1
1		1														31		32
2			1												15		32	496
3				1										7		24		4960
4					1								3		16		24	34720
4					1								7				48	1240
5						1						1		10		20		166656
5						1						3		4		24		34720
6							1				0		6		15		20	444416
6							1				1		4		14		24	416640
6							1				1		0		30		0	27776
6							1				3		0		12		32	17360
7								1		1		2		6		22		416640
7								1		1		0		12		18		277760
7								1		3		0		0		28		4960
7								1		0		3		9		19		2222080
7								1		0		1		15		15		444416
8									1		1		6		15		18	4444160
8									1		2		4		14		22	3333120
8									1		0		8		16		14	1666560
8									1		0		12		0		38	277760
8									2		0		4		16		20	312480
8									2		0		8		0		44	26040
8									2		2		0		14		28	59520
8									4		0		0		0		56	155
9										1		3		13		15		8888320
9										1		5		7		19		6666240
9										2		2		10		18		4999680
9										2		4		4		22		416640
9										3		1		7		21		317440
10											1		10		15		12	2666496
10											2		8		14		16	10416000
10											3		6		13		20	6666240
10											4		0		28		0	496000
10											4		4		12		24	1805440
10											6		0		10		32	13888
11												4		16		12		1666560
11												6		10		16		7027328
12													12		16		8	208320
12													16		0		32	219604

Remark 1 *In forming the table major tools are (ii) and (iv) stated in Section 10.*

12 A summary of known results (not exhaustive)

An illustrative table of the determination of the covering radius and the complete coset weight distribution

$r\backslash m$	3	4	5	6	7	8	9
1 1	$\rho=2$ **CWD**	$\rho=6$ **CWD**	$\rho=12$ **CWD**	$\rho=28$ **CWD**	$\rho=56$ **CWD**	$\rho=120$ *hard*	$240\le\rho\le 244$ *hard*
2 2	$\rho=1$	$\rho=2$ **CWD**	$\rho=6$ **CWD**	$\rho=18$ $CWD\ p.$	$40\le\rho\le 44$ $CWD\ hard$	$84\le\rho\le 100$ $CWD\ hard$	$171\le\rho\le 220$ $CWD\ hard$
3 3		$\rho=1$ **CWD**	$\rho=2$ **CWD**	$\rho=8$ $CWD\ p.$	$20\le\rho\le 23$ $CWD\ hard$	$43\le\rho\le 67$ $CWD\ hard$	$134\le\rho\le 167$ $CWD\ hard$
4 4			$\rho=1$ **CWD**	$\rho=2$ $CWD\ p.$	$\rho=8$ $CWD\ p.$	$22\le\rho\le 31$ $CWD\ hard$	$62\le\rho\le 98$ $CWD\ hard$
5 5				$\rho=1$	$\rho=2$ $CWD\ p.$	$\rho=10$ $CWD\ hard$	$23\le\rho\le 41$ $CWD\ hard$

$\rho=\rho(r,m)$: the covering radius of $R(r,m)$
CWD : Complete coset weight distribution is determined.
CWD p. : It is possible to determine the complete coset weight distribution
References : [5],[32],[33],[43],[45],[51],[59],[60].

13 BCH codes

For the definitions on BCH codes one may refer [41], Chap. 3 and Chap. 9.
Distribution of weights in the cosets of the BCH code
of length 15

generator matrix of this code

$$\begin{pmatrix}
1&0&0&0&1&0&1&1&1&0&0&0&0&0&0\\
0&1&0&0&0&1&0&1&1&1&0&0&0&0&0\\
0&0&1&0&0&0&1&0&1&1&1&0&0&0&0\\
0&0&0&1&0&0&0&1&0&1&1&1&0&0&0\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&0\\
0&0&0&0&0&1&0&0&0&1&0&1&1&1&0\\
0&0&0&0&0&0&1&0&0&0&1&0&1&1&1
\end{pmatrix}$$

Inhomogeneous Jacobi polynomials for the above code are computed below.

(0) Jacobi polynomial of index 0 (= weight enumerator) for this BCH

$$\begin{aligned}
Jac(BCH_{15},\mathbf{v}_0|X,Z) &= 1+18X^5+30X^6+15X^7\\
&\quad +15X^8+30X^9+18X^{10}+X^{15}
\end{aligned}$$

(1) Jacobi polynomial of index 1 for this BCH. There are 15 reference vectors of weight 1.

$$
\begin{aligned}
&Jac(BCH_{15}, \mathbf{v}_1|X, Z) = \\
&\quad 1 + X^5(6Z + 12) + X^6(12Z + 18) + X^7(7Z + 8) \\
&\quad +X^8(8Z + 7) + X^9(18Z + 12) \\
&\quad +X^{10}(12Z + 6) + X^{15}Z
\end{aligned}
$$

(2) Jacobi polynomial of index 2.
(2)-(i) There arise 90 this type of vectors.

$$
\begin{aligned}
&Jac^{(1)}(BCH_{15}, \mathbf{v}_2|X, Z) = \\
&\quad 1 + X^5(2Z^2 + 8Z + 8) + X^6(4Z^2 + 16Z + 10) + X^7(3Z^2 + 8Z + 4) \\
&\quad +X^8(4Z^2 + 8Z + 3) + X^9(10Z^2 + 16Z + 4) \\
&\quad +X^{10}(8Z^2 + 8Z + 2) + X^{15}Z^2
\end{aligned}
$$

(2)-(ii) There arise 15 this type of vectors.

$$
\begin{aligned}
&Jac^{(2)}(BCH_{15}, \mathbf{v}_2|X, Z) = \\
&\quad 1 + X^5(12Z + 6) + X^6(6Z^2 + 12Z + 12) + X^7(3Z^2 + 8Z + 4) \\
&\quad +X^8(4Z^2 + 8Z + 3) + X^9(12Z^2 + 12Z + 6) \\
&\quad +X^{10}(6Z^2 + 12Z) + X^{15}Z^2
\end{aligned}
$$

(3) Jacobi polynomial of index 3. There 4 kinds of rigid polynomials.
(3)-(i) There arise 60 this type of vectors.

$$
\begin{aligned}
&Jac^{(1)}(BCH_{15}, \mathbf{v}_3|X, Z) = \\
&\quad 1 + X^5(6Z^2 + 6Z + 6) + X^6(12Z^2 + 12Z + 6) \\
&\quad +X^7(Z^3 + 6Z^2 + 6Z + 2) + X^8(2Z^3 + 6Z^2 + 6Z + 1) \\
&\quad +X^9(6Z^3 + 12Z^2 + 12Z) + X^{10}(6Z^3 + 6Z^2 + 6Z) + X^{15}Z^3
\end{aligned}
$$

(3)-(ii) There arise 5 this type of vectors.

$$
\begin{aligned}
&Jac^{(2)}(BCH_{15}, \mathbf{v}_3|X, Z) = \\
&\quad 1 + X^5(6Z^2 + 6Z + 6) + X^6(2Z^3 + 6Z^2 + 18Z + 4) \\
&\quad +X^7(3Z^3 + 12Z) + X^8(12Z^2 + 3) + \\
&\quad X^9(4Z^3 + 18Z^2 + 6Z + 2) + X^{10}(6Z^3 + 6Z^2 + 6Z) + X^{15}Z^3
\end{aligned}
$$

(3)-(iii) There arise 180 this type of vectors.

$$
\begin{aligned}
&Jac^{(3)}(BCH_{15}, \mathbf{v}_3|X, Z) = \\
&\quad 1 + X^5(4Z^2 + 10Z + 4) + X^6(2Z^3 + 8Z^2 + 14Z + 6) \\
&\quad +X^7(Z^3 + 6Z^2 + 6Z + 2) + X^8(2Z^3 + 6Z^2 + 6Z + 1) \\
&\quad +X^9(6Z^3 + 14Z^2 + 8Z + 2) + X^{10}(4Z^3 + 10Z^2 + 4Z) + X^{15}Z^3
\end{aligned}
$$

(3)-(iv) There arise 30 this type of vectors.

$$Jac^{(4)}(BCH_{15}, \mathbf{v}_3|X, Z) =$$
$$1 + X^5(18Z) + X^6(18Z^2 + 12) + X^7(3Z^3 + 12Z)$$
$$+X^8(12Z^2 + 3) + X^9(12Z^3 + 18Z) + X^{10}(18Z^2) + X^{15}Z^3$$

The above computation leads to ,via Theorems 7 and 8, to the complete coset weight distribution below.

i coset weight	0	1	2	3	4	5	6	7	num. of diff. cosets
0	1	0	0	0	0	18	30	15	1
1		1	0	0	6	12	19	26	15
2			1	2	4	11	20	26	90
2			1	0	6	15	16	26	15
3				1	3	18	30	12	5
3				1	7	14	18	24	60
3				3	5	10	22	24	60
3				3	9	6	10	36	10

e	1	2	3	≥ 4
covering radius contributer	1 not specified	3 G-P-Z	5 H-B, D, H	unknown none
coset wt distrib.		C-C-M, C	unknown	unknown

Explanations: G-P-Z [24], H-B [31], D [22], H [27], C-C-M [9], C [11].

Remark 2 *Since making of our table was done around the year 1995, the present records may be out of date in some respects. Intensive readers may search updated results by using MathSci at the internet.*

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