A bijective CPS-translation between classical and intuitionistic proofs
– Extended Abstract –

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Abstract. We show that the novel CPS-translation due to Hofmann-Streicher is complete for second order $\lambda\mu$-calculus. For this, we introduce a universe of the translation, which is closed under reduction rules, and also a context-free grammar that describes the universe. Then it is syntactically proved that the CPS-translation is complete by defining an inverse translation from the universe. The inverse translation, in fact, forms a bijection between the universe and $\lambda\mu$-calculus, i.e., intuitionistic and classical proofs. Moreover, this work on second order typed case reveals interesting connections to the de Morgan's duality and also a non-trivial subsystem of the system **F** of Girard as a target calculus.

1 Introduction

The term CPS-translation, in general, denotes a program translation method into *c*ontinuation-*p*assing *s*tyle that is the meaning of the program as a function taking the rest of the program. The method has been studied for program transformation, definitional interpreter, or denotational semantics [14].

On the other hand, according to Griffin [5], a CPS-translation corresponds to a logical embedding from classical logic into intuitionistic logic under the Formulae-as-Types correspondence [7]. Parigot [10–12] introduced the $\lambda\mu$-calculus from the viewpoint of classical logic, and established an extension of the Curry-Howard isomorphism [5, 8]. Semantics of $\lambda\mu$-calculus has been investigated recently from the viewpoint of continuations. There have been noteworthy investigations including Hofmann-Streicher [6], Streicher-Reus [16], and Selinger [15]: In terms of a category of continuations, a continuation semantics of simply typed $\lambda\mu$-calculus is proved to be sound and complete for any $\lambda\mu$-theory [6]. Under the control category, it is established that an isomorphism between call-by-name and call-by-value $\lambda\mu$-calculi with conjunction and disjunction types [15]. The category of negated domains is introduced as a model of type free $\lambda\mu$-calculus [16]. Streicher-Reus also remarked that a CPS-translation naïvely based on Plotkin [13] cannot validate (η)-rule. All of the work involve a novel CPS-translation which requires, at least, products as a primitive notion, so that the extensionality, (η)-rule can be validated by the surjective pairing, as observed in [2].

As far as we know, however, polymorphic $\lambda\mu$-calculus has less been studied from the viewpoint of continuations. In this paper, we show that the novel CPS-translation is complete for second order $\lambda\mu$-calculus that gives a proof code of second order classical logic. For this, we introduce a universe of the translation, which is closed under reduction rules, and also a context-free grammar that describes the universe. Then by defining an inverse translation from the universe, it is syntactically proved that the CPS-translation is sound and complete for second order $\lambda\mu$-calculus with (η)-rule. The inverse translation, in fact, forms a bijection between the universe and $\lambda\mu$-calculus, in other words, intuitionistic proofs and classical ones. This paper contains a non-trivial extension of the technique in [3], which proved that the CPS-translation is complete for type free $\lambda\mu$-calculus. Moreover, this extension to second order typed case reveals interesting connections to the de Morgan's duality together with introduction-elimination duality like in Selinger [15] or Wadler [18] and also to a non-trivial subsystem of the system **F** of Girard as a target calculus.

2 Second order typed $\lambda\mu$-calculus

We give a definition of typed $\lambda\mu$-calculus *à la* Church as second order classical logic and, in other words, explicitly typed polymorphic calculus or the system **F** with control operator, denoted by $\lambda 2\mu$, see also Parigot [10–12]. The syntax of types is defined from $\bot$ (falsum) and type variables denoted by X, using $\Rightarrow$ or $\forall$ over type variables. The syntax of $\lambda\mu$-terms is from individual variables denoted by x and names denoted by α, using term-applications, type-applications, named terms $[\alpha]M$, λ-abstractions over individual variables or type variables, or μ-abstractions over names.

Definition 1 (Types).

$$A ::= \bot \mid X \mid A \Rightarrow A \mid \forall X.A$$
$$\neg A \;\equiv\; A \Rightarrow \bot$$

Definition 2 (Pseudo-terms).

$$\Lambda 2\mu \ni M ::= x \mid \lambda x\!:\!A.M \mid MM \mid \lambda X.M \mid MA \mid \mu\alpha\!:\!A.M \mid [\alpha]M$$

Definition 3 (Reduction rules).

(β) $(\lambda x\!:\!A.M)M_1 \to M[x := M_1]$
(η) $\lambda x\!:\!A.Mx \to M$ *if* $x \notin FV(M)$
(β_t) $(\lambda X.M)A \to M[X := A]$
(μ) $(\mu\alpha\!:\!(A_1 \Rightarrow A_2).M)M_1 \to \mu\alpha\!:\!A_2.M[\alpha \Leftarrow M_1]$
(μ_t) $(\mu\alpha\!:\!(\forall X.A).M)A_1 \to \mu\alpha\!:\!(A[X := A_1]).M[\alpha \Leftarrow A_1]$
(μ_β) $[\beta](\mu\alpha\!:\!A.M) \to M[\alpha := \beta]$
(μ_η) $\mu\alpha\!:\!A.[\alpha]M \to M$ *if* $\alpha \notin FN(M)$
$(\mu_\bot)$ $\mu\alpha\!:\!\bot.M \to M$ *and* $[\alpha]M \to M$ *if* α *with type* $\bot$

$FV(M)$ or $FN(M)$ stands for the set of free variables or names respectively in M. The term $M[\alpha \Leftarrow U]$ where U is a term or a type denotes a term obtained by replacing each subterm of the form $[\alpha]M'$ in M with $[\alpha](M'U)$. The one step reduction relation is denoted by $\to_R$ where R is (β), (η), etc. We write $\to_R^+$ or $\to_R^*$ to denote the transitive closure or the reflexive and transitive closure of $\to_R$, respectively. We employ the notation $=_{\lambda\mu}$ for the reflexive and transitive closure of the one step reduction $\to_{\lambda\mu}$ defined above.

The typing judgement of $\lambda 2\mu$ takes the form of $\Gamma \vdash M : A \mid \Delta$, where Γ is a set of a declaration in the form of $x : A$ with distinct variables as subjects, and Δ is a set of a declaration in the form of $\alpha : A$ with distinct names (continuation variables) as subjects and types as predicates.

Definition 4 (Type assignment rules).

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A \mid \Delta}$$

$$\frac{\Gamma, x : A_1 \vdash M : A_2 \mid \Delta}{\Gamma \vdash \lambda x : A_1.M : A_1 \Rightarrow A_2 \mid \Delta}\ (\Rightarrow I) \qquad \frac{\Gamma \vdash M_1 : A_1 \Rightarrow A_2 \mid \Delta \quad \Gamma \vdash M_2 : A_1 \mid \Delta}{\Gamma \vdash M_1 M_2 : A_2 \mid \Delta}\ (\Rightarrow E)$$

$$\frac{\Gamma \vdash M : A \mid \Delta}{\Gamma \vdash \lambda X.M : \forall X.A \mid \Delta}\ (\forall I)^* \qquad \frac{\Gamma \vdash M : \forall X.A \mid \Delta}{\Gamma \vdash M A_1 : A[X := A_1] \mid \Delta}\ (\forall E)$$

$$\frac{\Gamma \vdash M : \bot \mid \alpha : A, \Delta}{\Gamma \vdash \mu\alpha : A.M : A \mid \Delta}\ (\bot E) \qquad \frac{\Gamma \vdash M : A \mid \Delta}{\Gamma \vdash [\alpha]M : \bot \mid \Delta, \alpha : A}\ (\bot I)$$

where $(\forall I)^*$ *denotes the eigenvariable condition.*

The typing rule above is a formalization of classical logic, in the sense that we have $x_1 : A_1, \ldots, x_m : A_m \vdash_{\lambda 2\mu} M : A \mid \alpha_1 : B_1, \ldots, \alpha_n : B_n$ for some M iff $A_1, \ldots, A_m; \neg B_1, \ldots, \neg B_n \vdash A$ is provable in second order classical logic. Under this reading, $(\bot E)$ plays a rôle of reductio ad absurdum rule.

It is remarked that the following equation (ζ) is derivable from (η) and (μ) in the equational theory induced by $\to_{\lambda\mu}$:

(ζ): $\mu\alpha : (A_1 \Rightarrow A_2).M = \lambda x : A_1.\mu\alpha : A_2.M[\alpha \Leftarrow x]$

where x is a fresh variable.

This rule with a side condition is called (S_3) in [11]. In particular, the rule (μ_ζ) in [9, 6] is derivable from $(\mu_\bot)$.

(μ_ζ): $\mu\alpha : \neg A.M = \lambda x : A.M[\alpha \Leftarrow^* x]$ where x is a fresh variable.

Here, $M[\alpha \Leftarrow^* M_1]$ is a term obtained by replacing each subterm $[\alpha]M'$ in M with $M'M_1$.

3 Target calculus: intuitionistic logic with second order weak sum

Next we introduce our target calculus $\lambda^{\exists}$ as second order intuitionistic logic consisting of $\Rightarrow$, $\wedge$ and second order weak sums.

Definition 5 (Types).

$$A ::= \bot \mid X \mid A \Rightarrow A \mid A \wedge A \mid \exists X.A$$

Definition 6 (Pseudo-terms).

$$\begin{aligned}\Lambda^{\exists} \ni M ::= x \mid \lambda x\!:\!A.M \mid MM \mid \langle M, M\rangle \mid \pi_1(M) \mid \pi_2(M)\\ \mid [A, M] \mid \mathtt{split}(M, \lambda X.\lambda x\!:\!A.M)\end{aligned}$$

Definition 7 (Reduction rules).

(β) $(\lambda x\!:\!A.M)M_1 \rightarrow M[x := M_1]$
(η) $\lambda x\!:\!A.Mx \rightarrow M$ *if* $x \notin FV(M)$
(π) $\pi_i\langle M_1, M_2\rangle \rightarrow M_i$ *for* $i = 1, 2$
(sp) $\langle \pi_1 M, \pi_2 M\rangle \rightarrow M$
($\mathtt{split}$) $\mathtt{split}([A_1, M_1], \lambda X.\lambda x\!:\!A.M) \rightarrow M[X := A_1, x := M_1]$

We write $=_{\lambda^{\exists}}$ for the reflexive and transitive closure of the one step reduction $\rightarrow_{\lambda^{\exists}}$ defined above.

Definition 8 (Type assignment rules).

$$\frac{x\!:\!A \in \Gamma}{\Gamma \vdash x : A}$$

$$\frac{\Gamma, x\!:\!A_1 \vdash M : A_2}{\Gamma \vdash \lambda x\!:\!A_1.M : A_1 \Rightarrow A_2}\ (\Rightarrow I) \qquad \frac{\Gamma \vdash M_1 : A_1 \Rightarrow A_2 \quad \Gamma \vdash M_2 : A_1}{\Gamma \vdash M_1 M_2 : A_2}\ (\Rightarrow E)$$

$$\frac{\Gamma \vdash M_1 : A_1 \quad \Gamma \vdash M_2 : A_2}{\Gamma \vdash \langle M_1, M_2\rangle : A_1 \wedge A_2}\ (\wedge I) \qquad \frac{\Gamma \vdash M : A_1 \wedge A_2}{\Gamma \vdash \pi_i M : A_i}\ (\wedge E)$$

$$\frac{\Gamma \vdash M : A[X := A_1]}{\Gamma \vdash [A_1, M] : \exists X.A}\ (\exists I) \qquad \frac{\Gamma \vdash M : \exists X.A \quad \Gamma, x\!:\!A \vdash M_1 : A_1}{\Gamma \vdash \mathtt{split}(M, \lambda X.\lambda x\!:\!A.M_1) : A_1}\ (\exists E)^*$$

where $(\exists E)^*$ *denotes the eigenvariable condition.*

4 CPS-translation and soundness

For a CPS-translation of proof terms from $\Lambda 2\mu$ into $\Lambda^{\exists}$, we define embedding of types, denoted by A^k and denotation of types, denoted by $A^\bullet$, so that the definition of CPS-translation gives denotation of proof terms to second order classical logic, in terms of λ-terms with second order weak sums.

Definition 9 (Embedding of types).

(1) $\bot^* = \neg\bot$
(2) $X^* = \neg X$
(3) $(A_1 \Rightarrow A_2)^* = (A_1^k \wedge A_2^*)$
(4) $(\forall X.A)^* = \exists X.A^*$
(5) $A^k = \neg A^*$

Let A^{kol} be the well-known Kolmogorov's negative translation of A. Then A^k is equivalent to A^{kol} in intuitionistic logic. However, if we had adopted A^{kol} instead of A^k, then the extensionality (η) could not be valid computationally [2] under the forthcoming interpretation.

Definition 10 (Denotation of types).

(1) $\bot^\bullet = \bot$
(2) $X^\bullet = X$
(3) $\neg((A_1 \Rightarrow A_2)^\bullet) = \neg\neg(A^\bullet) \wedge \neg(B^\bullet)$
(4) $\neg((\forall X.A)^\bullet) = \exists X.\neg(A^\bullet)$

The above definition intuitively says that $(A_1 \Rightarrow A_2)^\bullet$ is classically equivalent to $\neg A^\bullet \vee B^\bullet$, and $(\forall X.A)^\bullet$ to $\neg\exists X.\neg A^\bullet$.

Lemma 1. **(1)** *We have that* $\neg A^\bullet = A^*$ *and* $\neg\neg A^\bullet = A^k$.
We have that $A^\bullet[X := A_1^\bullet] = (A[X := A_1])^\bullet$.
(2) $(A[X := A_1])^* = A^*[X := A_1^\bullet]$ *and* $(A[X := A_1])^k = A^k[X := A_1^\bullet]$.

Proof. (1) By induction on the structure of A.
(2) By induction on the structure of A with (1) above. □

Let `id` be $\lambda x\!:\!\bot.x$ with type $\bot^*$.

Definition 11 (Denotation of proof terms).

(i) $[\![x]\!] = x$ *if* $\Gamma \vdash x : A \mid \Delta$
(ii) $[\![\lambda x\!:\!A_1.M]\!] = \lambda a\!:\!(A_1 \Rightarrow A_2)^*.(\lambda x\!:\!A_1^k.[\![M]\!])(\pi_1 a)(\pi_2 a)$
if $\Gamma \vdash \lambda x\!:\!A_1.M : A_1 \Rightarrow A_2 \mid \Delta$
(iii) $[\![M_1 M_2]\!] = \lambda a\!:\!A^*.[\![M_1]\!]\langle[\![M_2]\!], a\rangle$ *if* $\Gamma \vdash M_1 M_2 : A \mid \Delta$
(iv) $[\![\lambda X.M]\!] = \lambda a\!:\!(\forall X.A)^*.\mathtt{split}(a, \lambda X.\lambda a_1\!:\!A^*.[\![M]\!]a_1)$
if $\Gamma \vdash \lambda X.M : \forall X.A \mid \Delta$
(v) $[\![MA_1]\!] = \lambda a\!:\!(A[X := A_1])^*.[\![M]\!][A_1^\bullet, a]$ *if* $\Gamma \vdash MA_1 : A[X := A_1] \mid \Delta$
(vi) $[\![\mu a\!:\!A.M]\!] = \lambda a\!:\!A^*.[\![M]\!]\mathtt{id}$ *if* $\Gamma \vdash \mu a\!:\!A.M : A \mid \Delta$
(vii) $[\![[a]M]\!] = \lambda k\!:\!\bot^*.[\![M]\!]a$ *if* $\Gamma \vdash [a]M : \bot \mid \Delta, a\!:\!A$

The definition above interprets each proof term with type A as a functional element with type A^k (space of denotations of type A), which takes, as an argument, a continuation with type A^*. The cases of application say that continuations are in the form of a pair $\langle [\![M]\!], a\rangle$ or $[A^\bullet, a]$ consisting of a denotation and a continuation in this order. The cases of λ-abstraction mean that after the interpretation λ-abstraction is waiting for a first component of a continuation (i.e., a denotation of its argument), and the second component becomes a rest continuation to the result. Here, in fact, we have that $(A_1 \Rightarrow A_2)^* = A_1^k \wedge A_2^*$ and $(\forall X.A)^* = \exists X.A^*$. Finally, names of $\lambda\mu$-calculus are simply interpreted as continuation variables. It is also remarked that $(\Rightarrow I)$ and $(\Rightarrow E)$ are respectively interpreted by $(\wedge E)$ and $(\wedge I)$, and $(\forall I)$ and $(\forall E)$ are respectively by $(\exists E)$ and $(\exists I)$, strictly speaking intuitionistic ones. Moreover, $\bot^*$ behaves like a singleton type *à la* truth. The de Morgan like duality * together with introduction-elimination duality has connections to Selinger [15] and Wadler [18].

Example 1. (1) Let $xM_1 \cdots M_n : A$ and $m, n \geq 0$:

$$
\begin{aligned}
&[\![\lambda x_1 \!:\! A_1 \ldots \lambda x_m \!:\! A_m . x M_1 \cdots M_n]\!] \\
&\to_\beta^+ \lambda a \!:\! (A_1^k \wedge \cdots \wedge A_m^k \wedge A^*). x \langle [\![M_1]\!], \ldots, \langle [\![M_n]\!], \pi_2^m a\rangle \ldots \rangle \\
&\qquad\qquad\qquad [x_1 := \pi_1 a, x_2 := \pi_1(\pi_2 a), \ldots, x_m := \pi_1(\pi_2^{m-1} a)]
\end{aligned}
$$

(2) An example of Peirce's formula, $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$:

$$
\begin{aligned}
&[\![\lambda y \!:\! ((A \Rightarrow B) \Rightarrow A). \mu a \!:\! A.[a](y(\lambda x \!:\! A. \mu b \!:\! B.[a]x))]\!] \\
&\to_\beta^+ \lambda a \!:\! (((A \Rightarrow B) \Rightarrow A) \Rightarrow A)^*. \pi_1 a \langle \lambda b \!:\! (A \Rightarrow B)^*. \pi_1 b(\pi_2 a), \pi_2 a\rangle
\end{aligned}
$$

We need some axioms for handling $\bot^*$ as the unit type in the extensional theory of $\lambda^\exists$.

Definition 12 (Axiom for $\mathtt{id} : \bot^*$). *The term* $\mathtt{id}$ *is the canonical inhabitant of the unit type* $\bot^*$*, i.e.,* $\mathtt{id} = k$ *for any* $k \in \mathcal{K}$ *with type* $\bot^*$ *where* $\mathcal{K}$ *will be defined soon.*

Lemma 2. (1) *For well-typed* $M[\alpha \Leftarrow M_1] \in \Lambda 2\mu$*, we have* $[\![M[\alpha \Leftarrow M_1]]\!] \to_\beta^* [\![M]\!][\alpha := \langle [\![M_1]\!], \alpha\rangle]$.
(2) *For well-typed* $M[\alpha \Leftarrow A_1] \in \Lambda 2\mu$*, we have* $[\![M[\alpha \Leftarrow A_1]]\!] \to_\beta^* [\![M]\!][\alpha := [A_1^\bullet, \alpha]]$.

Proof. By induction on the strcture of M. □

We write $\Gamma \vdash M_1 =_{\lambda\mu} M_2 : A \mid \Delta$ for $\begin{cases} \Gamma \vdash M_1 : A \mid \Delta, \\ \Gamma \vdash M_2 : A \mid \Delta, \text{ and} \\ M_1 =_{\lambda\mu} M_2. \end{cases}$

And similarly to the case of $\lambda^\exists$.

Proposition 1 (Soundness). *If we have* $\Gamma \vdash M_1 =_{\lambda\mu} M_2 : A \mid \Delta$*, then* $\Gamma^k, \Delta^* \vdash [\![M_1]\!] =_{\lambda^\exists} [\![M_2]\!] : A^k$.

Proof. If we have $\Gamma \vdash M : A \mid \Delta$, then $\Gamma^k, \Delta^* \vdash [\![M]\!] : A^k$ by induction on the derivation together with Definition 11. We show one case of $\lambda X.M : \forall X.A$. Suppose the following figure of $\lambda 2\mu$, where X is not free in any assumption on which A depends.

$$\frac{M : A}{\lambda X.M : \forall X.A}\ (\forall I)^*$$

Then we have the proof figure of $\lambda^\exists$, where the eigenvariable condition of $(\exists E)$ can be guaranteed by that of $(\forall I)$.

$$\frac{\dfrac{[a : \exists X.A^*] \qquad \dfrac{[\![M]\!] : A^k \quad [a_1 : A^*]}{[\![M]\!]a_1 : \bot}}{\mathtt{split}(a, \lambda X.\lambda a_1 \!:\! A^*.[\![M]\!]a_1) : \bot}\ (\exists E)^*}{\lambda a \!:\! (\forall X.A)^*.\mathtt{split}(a, \lambda X.\lambda a_1 \!:\! A^*.[\![M]\!]a_1) : (\forall X.A)^k}$$

Next we can prove that if we have $M_1 =_{\lambda\mu} M_2$ then $[\![M_1]\!] =_{\lambda^\exists} [\![M_2]\!]$ by induction on the derivation for well-typed terms. We show one case of (β_t). Let $\lambda X.M : \forall X.A$.

$$\begin{aligned}
&[\![(\lambda X.M)A_1]\!] \\
&\quad = \ \lambda a \!:\! (A[X := A_1])^*.(\lambda b \!:\! (\forall X.A)^*.\mathtt{split}(b, \lambda X.\lambda c \!:\! A^*.[\![M]\!]c))[A_1^\bullet, a] \\
&\ \to_\beta \ \lambda a \!:\! (A[X := A_1])^*.\mathtt{split}([A_1^\bullet, a], \lambda X.\lambda c \!:\! A^*.[\![M]\!]c) \\
&\ \to_{\lambda^\exists} \ \lambda a \!:\! (A[X := A_1])^*.[\![M]\!][X := A_1^\bullet]a \\
&\quad = \ \lambda a \!:\! (A[X := A_1])^*.[\![M[X := A_1]]\!]a \ \text{ from Lemma 1 (2)} \\
&\ \to_\eta \ [\![M[X := A_1]]\!]
\end{aligned}$$

□

5 Inverse translation and completeness

We introduce a context-free grammar to describe the image of the CPS-translation closed under the reduction rules.

Definition 13 (CPS-grammar).

$$\begin{aligned}
\mathcal{R} &::= x \mid \pi_1\mathcal{K} \mid (\lambda x \!:\! A^k.\mathcal{R})\mathcal{R}_1 \mid \lambda a \!:\! A^*.\mathcal{W} \\
\mathcal{K} &::= a \mid \mathtt{id} \mid \pi_2\mathcal{K} \mid \langle \mathcal{R}, \mathcal{K} \rangle \mid [A^\bullet, \mathcal{K}] \\
\mathcal{W} &::= \mathcal{R}\mathcal{K} \mid \mathtt{split}(\mathcal{K}, \lambda X.\lambda a \!:\! A^*.\mathcal{W})
\end{aligned}$$

Next we give a refinement of Definition 7 for well-typed $R \in \mathcal{R}$, $K \in \mathcal{K}$ and $W \in \mathcal{W}$.

Definition 14. (β_x) $(\lambda x \!:\! A^k.R)R_1 \to R[x := R_1]$
(β_a) $(\lambda a \!:\! A^*.W)K \to W[a := K]$
(η_a) $\lambda a \!:\! A^*.Ra \to R$
$(\exists_{R,K})$ $\mathtt{split}([A_1^\bullet, K], \lambda X.\lambda a \!:\! A^*.W) \to W[X := A_1^\bullet, a := K]$
$(\pi_{R,K})$ $\pi_1\langle R, K\rangle \to R$ *and* $\pi_2\langle R, K\rangle \to K$

(sp_K) $\langle \pi_1 K, \pi_2 K\rangle \to K$

Lemma 3 (Subject reduction w.r.t. $\mathcal{R}$, $\mathcal{K}$ and $\mathcal{W}$). *$\mathcal{R}$, $\mathcal{K}$ and $\mathcal{W}$ are closed under the reduction rules.*

Proof. Substitutions associated to the reduction rules are closed with respect to the categories $\mathcal{R}$, $\mathcal{K}$, and $\mathcal{W}$. □

Lemma 4 (Subject reduction w.r.t. types). *If we have $R : A^k$, $K : A^*$ and $W : \bot$ together with $R \to_{\lambda^\exists} R_1$, $K \to_{\lambda^\exists} K_1$ and $W \to_{\lambda^\exists} W_1$, then $R_1 : A^k$, $K_1 : A^*$ and $W_1 : \bot$.*

Proof. The calculus $\lambda^\exists$ has the subject reduction property for well-typed terms. □

Definition 15 (Universe of the CPS-translation).

$$Univ_{2\mu} \stackrel{\text{def}}{=} \{P \in \Lambda^\exists \mid [\![M]\!] \to^*_{\lambda^\exists} P \textit{ for some } \Gamma, \Delta, A \textit{ such that } \Gamma \vdash_{\lambda 2\mu} M : A \mid \Delta\}$$

Proposition 2. *$Univ_{2\mu}$ is generated by $\mathcal{R}$.*

Proof. For well typed $M \in \Lambda 2\mu$, we have $[\![M]\!] \in \mathcal{R}$, and moreover the categories are closed under the restricted reduction rules by Lemma 3. □

Lemma 5. *For any $P \in Univ_{2\mu}$, we have some Γ, Δ and A such that $\Gamma^k, \Delta^* \vdash_{\lambda^\exists} P : A^k$.*

Proof. From the definition of $Univ_{2\mu}$, Proposition 1 and Lemma 4. □

We simply write $A_1^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*$ for $A_1^k \wedge (\cdots \wedge (A_n^k \wedge A_{n+1}^*) \cdots)$ because of the right associativity of $\Rightarrow$. It is noted that

$$\begin{aligned}(A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow A_n \Rightarrow A_{n+1})^* &= A_1^k \wedge (A_2 \Rightarrow \cdots \Rightarrow A_n \Rightarrow A_{n+1})^* \\ &= A_1^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*\end{aligned}$$

We employ letters U, V for either a term $R \in \mathcal{R}$ or a type $A^\bullet$. We write (U, K) for either $\langle U, K\rangle$ or $[U, K]$, and $(U_1, \ldots, U_n, K)$ is an abbreviation of $(U_1, \ldots, (U_n, K) \ldots)$ for $n \geq 0$.

Following the context-free grammar, well-typed $K \in \mathcal{K}$ in projection normal form is one of the following forms:

$$K = \begin{cases} (U_1, \ldots, U_m, \mathtt{id}) & \text{where } \mathtt{id} : \bot^* \\ (U_1, \ldots, U_m, a) & \text{where } a : \bot^*,\ a : X^* \text{ or } a : \exists X.A^* \\ (U_1, \ldots, U_m, \pi_2^n a) & \text{where } a : A_1^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^* \text{ with } n \geq 1 \end{cases}$$

Throughout the paper, we consider projection normal forms. In addition, it is observed that types are closed under the applications of (β_a)-expansion and (sp_K)-expansion, in the following sense:

Suppose that $\pi_1(\pi_2^i a)$ where $0 \leq i \leq n-1$ appears in R or U_j $(1 \leq j \leq m)$, but there is no tail term $\pi_2^p b$ with b such that $b \equiv a$:

$$\lambda a\!:\!A^*.R(U_1,\ldots,U_m,\pi_2^p b) \in \mathcal{K} \ \text{ with } A^* = A_1^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*$$

Then we can apply (β_a)-expansion to produce the well-typed and corresponding $\underline{\text{tail } (\pi_2 a)}$ to the top λa with the same variable a:

$$\lambda a\!:\!A^*.(\lambda c\!:\!(A_2^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*).R(U_1,\ldots,U_m,\pi_2^p b))\underline{(\pi_2 a)} \in \mathcal{K}$$

where c is a fresh continuation variable with type $A_2^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*$. Moreover, $(\pi_2 a)$ can be (sp_K)-expanded to $\langle \pi_1(\pi_2 a), \pi_2^2 a\rangle : A_2^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*$, so that one can obtain the well-typed term $\langle \pi_1(\pi_2 a), \ldots, \langle \pi_1(\pi_2^{n-1} a), \pi_2^n a\rangle \ldots\rangle$.

Well-typed `split` is one of the following forms up to projections:

$$\begin{cases} \texttt{split}(\pi_2^n a, \lambda X.\lambda a_1\!:\!A^*.W) & \text{with } \pi_2^n a : \exists X.A^* \text{ for } n \geq 0 \\ \texttt{split}([A_1^\bullet, K], \lambda X.\lambda a_1\!:\!A^*.W) & \text{with } [A_1^\bullet, K] : \exists X.A^* \end{cases}$$

We are now ready to give the definition of the inverse translation $\sharp$ for elements in $\mathit{Univ}_{2\mu}$ under projections, (sp_K)-expansions and (β_a)-expansions.

Definition 16 (Inverse translation $\sharp$ for $\mathit{Univ}_{2\mu}$).

(i) $x^\sharp = x$
(ii) $\pi_1(\pi_2^i a)^\sharp = a_{i+1}$ *with* $0 \leq i \leq n-1$ *and* $a : A_1^k \wedge \cdots \wedge A_n^k \wedge A_{n+1}^*$, *where* a_{i+1} *is a fresh individual variable*
(iii) $((\lambda x\!:\!A^k.R)R_1)^\sharp = (\lambda x\!:\!A.R^\sharp)R_1^\sharp$
(iv) *Case* W *of* RK:

- **(1)** *Case of* $\perp$;
  $(\lambda a\!:\!\perp^*.W)^\sharp = W^\sharp$
- **(2)** *Case of* $A \not\equiv \perp$;
  $(\lambda a\!:\!A^*.W)^\sharp = \mu a\!:\!A.W^\sharp$

(v) *Case* W *of* `split`: *Let* x *be a fresh variable.*

$$\begin{aligned} &(\lambda a\!:\!A^*.\texttt{split}(K, \lambda X.\lambda c\!:\!C^*.W))^\sharp \\ &= \ \underline{\texttt{let}}\ x = \lambda X.(\lambda c\!:\!C^*.W)^\sharp\ \underline{\texttt{in}}\ (\lambda a\!:\!A^*.xK)^\sharp \end{aligned}$$

where $W^\sharp$ *is defined for* $W = RK$ $(m \geq 0, n \geq 1)$ *together with* $(A^\bullet)^\sharp = A$:

- **(1)** *Case of* $b : \perp^*$;
  $(R(U_1,\ldots,U_m,b))^\sharp = R^\sharp U_1^\sharp \ldots U_m^\sharp$
- **(2)** *Case of* $b : X^*$ *or* $b : \exists X.A^*$;
  $(R(U_1,\ldots,U_m,b))^\sharp = [b](R^\sharp U_1^\sharp \ldots U_m^\sharp)$
- **(3)** *Case of* $b : B_1^k \wedge \cdots \wedge B_n^k \wedge B_{n+1}^*$;
  $(R(U_1,\ldots,U_m,\pi_2^n b))^\sharp = [b](\lambda b_1\!:\!B_1 \ldots \lambda b_n\!:\!B_n.R^\sharp U_1^\sharp \ldots U_m^\sharp)$

It is remarked that in the case of (v) above, $\underline{\mathtt{let}}\ x = P\ \underline{\mathtt{in}}\ Q$ is an expression of a meta-language to describe a $\lambda\mu$-term which is obtained by replacing x in Q with P. For instance, the definition (v) says that if $K = [B^\bullet, (U_1, \ldots, U_m, \pi_2^p b)]$ with $b : B_1^k \wedge \cdots \wedge B_p^k \wedge B_{p+1}^*$ and $W = R(V_1, \ldots, V_n, \pi_2^q d)$ with $d : D_1^k \wedge \cdots \wedge D_q^k \wedge D_{q+1}^*$, then we have

$$\begin{aligned}&(\lambda a\!:\!A^*.\mathtt{split}([B^\bullet, (U_1, \ldots, U_m, \pi_2^p b)], \lambda X.\lambda c\!:\!C^*.R(V_1, \ldots, V_n, \pi_2^q d)))^\sharp\\ &\ = \mu a\!:\!A.[b](\lambda b_1\!:\!B_1 \ldots \lambda b_p\!:\!B_p.\\ &\qquad (\lambda X.\mu c\!:\!C.[d](\lambda d_1\!:\!D_1 \ldots \lambda d_q\!:\!D_q.R^\sharp V_1^\sharp \ldots V_n^\sharp))BU_1^\sharp \ldots U_m^\sharp),\end{aligned}$$

where $A \not\equiv \bot$ and $C \not\equiv \bot$.

Example 2. The example appeared in Example 1 (2) is decompiled under $\underline{(\beta_a)\text{-expansion}}$ with a fresh continuation variable c, as follows:

$$\begin{aligned}&(\lambda a\!:\!(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)^*.\pi_1 a\langle \underline{\lambda b\!:\!(A \Rightarrow B)^*.\pi_1 b(\pi_2 a)}, \pi_2 a\rangle)^\sharp\\ &= \ (\lambda a\!:\!(((A \Rightarrow B) \Rightarrow A) \Rightarrow A)^*.\pi_1 a\langle \underline{\lambda b\!:\!(A \Rightarrow B)^*.(\lambda c\!:\!B^*.\pi_1 b(\pi_2 a))(\pi_2 b)}, \pi_2 a\rangle)^\sharp\\ &= \ \mu a\!:\!(((A \Rightarrow B) \Rightarrow A) \Rightarrow A).[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).a_1\\ &\qquad (\mu b\!:\!(A \Rightarrow B).[b](\lambda b_1\!:\!A.\mu c\!:\!B.[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).b_1))))\\ &\to_{\mu_\eta} \mu a\!:\!(((A \Rightarrow B) \Rightarrow A) \Rightarrow A).[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).a_1\\ &\qquad (\lambda b_1\!:\!A.\mu c\!:\!B.[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).b_1)))\\ &=_{\lambda\mu} \lambda y\!:\!((A \Rightarrow B) \Rightarrow A).\mu a\!:\!A.[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).a_1\\ &\qquad (\lambda b_1\!:\!A.\mu c\!:\!B.[a](\lambda a_1\!:\!((A \Rightarrow B) \Rightarrow A).b_1)))[a \Leftarrow y]\\ &\to_\beta^+ \lambda y\!:\!((A \Rightarrow B) \Rightarrow A).\mu a\!:\!A.[a](y(\lambda b_1\!:\!A.\mu b\!:\!B.[a]b_1))\end{aligned}$$

Lemma 6. (1) *Let $M \in \Lambda 2\mu$ be a well-typed term. Then we have that $[\![M]\!]^\sharp \to_{\eta\mu_\eta}^* M$.*

(2) *Let $P \in \mathit{Univ}_{2\mu}$. Then we have that $[\![P^\sharp]\!]\theta \to_{\lambda^\exists}^* P$, where θ is a substitution for all free variables a_i in $[\![P^\sharp]\!]$ replaced by $\pi_1(\pi_2^{i-1} a)$.*

Proof. **(1)** By induction on the structure of well-typed $M \in \Lambda 2\mu$.
(2) By induction on the strucutre of $P \in \mathit{Univ}_{2\mu}$. □

Proposition 3 (Completeness1). *For $P \in \mathit{Univ}_{2\mu}$, if we have $\Gamma^k, \Delta^* \vdash P : A^k$ in $\lambda^\exists$, then $\Gamma \vdash P^\sharp : A \mid \Delta$ in $\lambda 2\mu$.*

Proof. By induction on the structure of $P \in \mathit{Univ}_{2\mu}$, which is a well-typed term of $\Lambda^\exists$.

In order to describe uniformly both $\Rightarrow$ and $\forall$, we emply the following notation Π for types A, B_i and for either a term or a type U_i:

$$\Pi U_i; B_i = \begin{cases} \mathit{nil} & \text{if } U_i \text{ is a type,}\\ U_i : B_i & \text{otherwise.}\end{cases}$$

And similarly

$$\Pi U_i; B_i.A = \begin{cases} \forall X_i.A & \text{if } U_i \text{ is a type,}\\ B_i \Rightarrow A & \text{otherwise.}\end{cases}$$

We show one case of $\lambda a : A^*.\mathtt{split}([C_1^\bullet, K], \lambda X.\lambda c : C^*.W)$. Let $A \not\equiv \bot$, $K \equiv (U_1, \ldots, U_m, \pi_2^p b)$, $C \not\equiv \bot$, and $W \equiv R(V_1, \ldots, V_n, \pi_2^q d)$ together with $b : B^* \equiv B_1^k \wedge \cdots \wedge B_p^k \wedge B_{p+1}^*$ and $d : D^* \equiv D_1^k \wedge \cdots \wedge D_q^k \wedge D_{q+1}^*$. Assume the following judgement of $\lambda^\exists$:

$$\Gamma^k, \Delta^* \vdash \lambda a{:}A^*.\mathtt{split}([B^\bullet, K], \lambda X.\lambda c{:}C^*.W) : A^k$$

Since $K : C^*[X := C_1^\bullet]$, we have the judgements of $\lambda^\exists$ for some type E_i ($1 \leq i \leq m$):

$$\Gamma^k, \Delta^*, a : A^* \vdash \Pi U_i; E_i^k$$

where $(C[X := C_1])^* \equiv (\Pi U_1; E_1 \ldots \Pi U_m; E_m.B_{p+1})^*$. Then $\pi_1(\pi_2^j b) : B_{j+1}^k$ in U_i can be replaced with $b_{j+1} : B_{j+1}^k$ ($0 \leq j \leq p-1$), whose result is denoted by U_i', so that we have the following judgement (2-1):

$$\Gamma^k, b_1 : B_1^k, \ldots, b_p : B_p^k, \Delta^*, a : A^*, b : B^* \vdash \Pi U_i'; E_i^k \quad \cdots \text{ (2-1)}$$

From the assumption, we also have the following judgements for some type F_i ($1 \leq i \leq n$):

$$\Gamma^k, \Delta^*, a : A^*, c : C^* \vdash R : (\Pi V_1; F_1 \ldots \Pi V_n; F_n.D_{q+1})^k$$

$$\Gamma^k, \Delta^*, c : C^* \vdash \Pi V_i; F_i^k$$

Then $\pi_1(\pi_2^j d) : D_{j+1}^k$ can also be replaced with $d_{j+1} : D_{j+1}^k$ ($0 \leq j \leq q-1$), so that one can obtain as follows:

$$\Gamma^k, d_1 : D_1^k, \ldots, d_q : D_q^k, \Delta^*, a : A^*, c : C^* \vdash R' : (\Pi V_1; F_1 \ldots \Pi V_n; F_n.D_{q+1})^k$$

$$\Gamma^k, d_1 : D_1^k, \ldots, d_q : D_q^k, \Delta^*, a : A^*, c : C^* \vdash \Pi V_i'; F_i^k$$

From the induction hypotheses, we have the judgements of $\lambda 2\mu$ as follows:

$$\Gamma, d_1 : D_1, \ldots, d_q : D_q \vdash R^\sharp : (\Pi V_1; F_1 \ldots \Pi V_n; F_n.D_{q+1}) \mid \Delta, a{:}A, c{:}C$$

$$\Gamma, d_1 : D_1, \ldots, d_q : D_q \vdash \Pi V_i^\sharp; F_i \mid \Delta, a{:}A, c{:}C$$

Hence the well-formed judgement of $\lambda 2\mu$ is derived as follows (2-2):

$$\Gamma \vdash (\lambda X.\mu c{:}C.[d](\lambda d_1{:}D_1 \ldots d_q{:}D_q.R^\sharp V_1^\sharp \ldots V_n^\sharp))C_1 : C[X := C_1] \mid \Delta, a{:}A$$

where the eigenvariable condition of $(\forall I)$ can be justified by that of $(\exists E)$.

On the other hand, the application of the induction hypotheses to (2-1) gives the following judgement of $\lambda 2\mu$:

$$\Gamma, b_1 : B_1, \ldots, b_p : B_p \vdash \Pi U_i^\sharp; E_i \mid \Delta, a{:}A, b{:}B$$

Since $C[X := C_1] \equiv \Pi U_1; E_1 \ldots \Pi U_m; E_m.B_{p+1}$, from (2-2) and the judgement above, we confirmed that the following judgement of $\lambda 2\mu$ for $P^\sharp$ is well-formed:

$$\Gamma \vdash P^\sharp : A \mid \Delta$$

where $P^\sharp \equiv \mu a{:}A.[b](\lambda b_1{:}B_1 \ldots b_p{:}B_p.$
$(\lambda X.\mu c{:}C.[d](\lambda d_1{:}D_1 \ldots d_q{:}D_q.R^\sharp V_1^\sharp \ldots V_n^\sharp))C_1 U_1^\sharp \ldots U_m^\sharp)$ □

Lemma 7. *Let $R[a := (U_1, \dots, U_m, \pi_2^n a)] \in \mathcal{R}$ be a well-typed term where U_i is either a term or a type, and $m, n \geq 0$.*

$$\begin{aligned}
&(R[a := (U_1, \dots, U_m, \pi_2^n a)])^\sharp [a \Leftarrow a_1, \dots, a_n] \\
&=_\beta R^\sharp [a \Leftarrow U_1^\sharp, \dots, U_m^\sharp] \\
&\qquad [a_1 := U_1^\sharp, \dots, a_m := U_m^\sharp, a_{m+1} := a_{n+1}, a_{m+2} := a_{n+2}, \dots]
\end{aligned}$$

provided that $U_1, \dots, U_m$ have no free occurrences of a.

It is remarked that if $U_i \equiv A^\bullet$ for some $1 \leq i \leq m$ and some type A, then we can assume that there is no free occurrence of a_i in $R^\sharp$. Suppose that for all $1 \leq j < i$, U_j is not a type, i.e., $U_j \equiv R_j$ for some term $R_j \in \mathcal{R}$. Then a in R has type $A_1^k \wedge \dots \wedge A_{j-1}^k \wedge \exists X.A_j^*$ for some types $A_1, \dots, A_j$. Hence, $\pi_1(\pi_2^l a)$ is well-typed only for $0 \leq l \leq j-2$, so that the inverse translation gives at most a_{j-1}. We also remark that since R can be $\lambda a\!:\!A^*.W$, the above lemma holds as well for $W \in \mathcal{W}$ instead of R.

Proof. By induction on the structure of well-typed R. We show an essential case of $\lambda b\!:\!B^*.R(\pi_2^q a)$ with $B \not\equiv \bot$, $q \geq 0$ and $b \not\equiv a$. Let θ, θ', Θ and Θ' be as follows:

$$\begin{aligned}
\theta &\equiv [a := K] \;\text{ with } K \equiv (U_1, \dots, U_m, \pi_2^n a) \text{ and } n \geq 0, \\
\theta' &\equiv [a_1 := U_1^\sharp, \dots, a_m := U_m^\sharp, a_{m+1} := a_{n+1}, a_{m+2} := a_{n+2}, \dots], \\
\Theta &\equiv [a \Leftarrow U_1^\sharp, \dots, U_m^\sharp], \;\text{ and} \\
\Theta' &\equiv [a \Leftarrow a_1, \dots, a_n].
\end{aligned}$$

Now we show that

$$(\lambda b\!:\!B^*.R(\pi_2^q a)\theta)^\sharp \Theta' =_\beta (\lambda b\!:\!B^*.R(\pi_2^q a))^\sharp \Theta\theta'.$$

Case 1. Every U_i $(1 \leq i \leq m)$ is a term:
Let $a : A_1^k \wedge \dots \wedge A_m^k \wedge A_{m+1}^*$. Then the right-hand side is rewritten as follows:

$$\begin{aligned}
&(\lambda b\!:\!B^*.R(\pi_2^q a))^\sharp \Theta\theta' \\
&= (\mu b\!:\!B.[a](\lambda a_1\!:\!A_1 \dots \lambda a_q\!:\!A_q.R^\sharp))\Theta\theta' \\
&= \mu b\!:\!B.[a](\lambda a_1\!:\!A_1 \dots \lambda a_q\!:\!A_q.R^\sharp\Theta)U_1^\sharp \cdots U_m^\sharp \theta'
\end{aligned}$$

Case 1-1. $q + 1 \leq m$:

$$\begin{aligned}
&\mu b\!:\!B.[a](\lambda a_1\!:\!A_1 \dots \lambda a_q\!:\!A_q.R^\sharp\Theta)U_1^\sharp \cdots U_m^\sharp \theta' \\
&\to_\beta^+ \mu b\!:\!B.[a](R^\sharp\Theta[a_1 := U_1^\sharp, \dots, a_q := U_q^\sharp]U_{q+1}^\sharp \cdots U_m^\sharp \theta') \\
&= \;\; \mu b\!:\!B.[a](R^\sharp\Theta\theta' U_{q+1}^\sharp \cdots U_m^\sharp) \\
&\qquad \text{since } U_i \text{ has no occurrences of } a \text{ and no free variables } a_j \; (j > q) \text{ appear} \\
&=_\beta \mu b\!:\!B.[a]((R\theta)^\sharp \Theta' U_{q+1}^\sharp \cdots U_m^\sharp) \\
&\qquad \text{by induction hypothesis}
\end{aligned}$$

$$\begin{aligned}
&=_\beta (\mu b{:}B.[a](\lambda a_1{:}A_1\ldots\lambda a_n{:}A_n.(R\theta)^\sharp U^\sharp_{q+1}\cdots U^\sharp_m))\Theta'\\
&= (\lambda b{:}B^*.R\theta(U_{q+1},\cdots,U_m,\pi_2^n a))^\sharp\Theta'\\
&= (\lambda b{:}B^*.R\theta(\pi_2^q(U_1,\cdots,U_m,\pi_2^n a)))^\sharp\Theta'\\
&= (\lambda b{:}B^*.R(\pi_2^q a)\theta)^\sharp\Theta'
\end{aligned}$$

Case 1-2. $q+1>m$:

$$\begin{aligned}
&\mu b{:}B.[a](\lambda a_1{:}A_1\ldots\lambda a_q{:}A_q.R^\sharp\Theta)U^\sharp_1\cdots U^\sharp_m\theta'\\
&\to^+_\beta \mu b{:}B.[a](\lambda a_{m+1}{:}A_{m+1}\ldots\lambda a_q{:}A_q.R^\sharp\Theta[a_1:=U^\sharp_1,\ldots,a_m:=U^\sharp_m]\theta')\\
&= \mu b{:}B.[a](\lambda a_{n+1}{:}A_{n+1}\ldots\lambda a_{q-m+n}{:}A_{q-m+n}.R^\sharp\Theta\theta')\\
&\quad\text{since } U_i \text{ has no occurrences of } a \text{ and no free variables } a_j\ (j\ge n+1) \text{ appear}\\
&=_\beta \mu b{:}B.[a](\lambda a_{n+1}{:}A_{n+1}\ldots\lambda a_{q-m+n}{:}A_{q-m+n}.(R\theta)^\sharp\Theta')\\
&\quad\text{by induction hypothesis}\\
&=_\beta (\mu b{:}B.[a](\lambda a_1{:}A_1\ldots\lambda a_{q-m+n}{:}A_{q-m+n}.(R\theta)^\sharp))\Theta'\\
&= (\lambda b{:}B^*.R\theta(\pi_2^{q-m+n}a))^\sharp\Theta'\\
&= (\lambda b{:}B^*.R\theta(\pi_2^q(U_1,\cdots,U_m,\pi_2^n a)))^\sharp\Theta'\\
&= (\lambda b{:}B^*.R(\pi_2^q a)\theta)^\sharp\Theta'
\end{aligned}$$

Case 2. U_j is a type for some $1\le j\le m$ and every U_i $(i<j)$ is a term:
We can assume that $a : A_1^k\wedge\cdots\wedge A^k_{j-1}\wedge\exists X.A^*_j$ and $q=j-1$ for $\pi_2^q a:\exists X.A^*_j$.
Then the equation to be proved becomes as follows:

$$(\lambda b{:}B^*.R(\pi_2^{j-1}a)\theta)^\sharp\Theta' =_\beta (\lambda b{:}B^*.R(\pi_2^{j-1}a))^\sharp\Theta\theta'$$

This case can be confirmed following the similar pattern to the above case 1-1 because of $j\le m$. □

Proposition 4 (Completeness2). *Let* $P,Q\in \mathit{Univ}_{2\mu}$.

(1) *If* $P\to_{\beta_x} Q$ *then* $P^\sharp\to_\beta Q^\sharp$.
(2) *If* $P\to_{\beta_a} Q$ *then* $P^\sharp =_{\lambda\mu} Q^\sharp$.
(3) *If* $P\to_{\eta_a} Q$ *then* $P^\sharp\to^+_{\eta\mu_\eta} Q^\sharp$.
(4) *If* $P\to_{\pi_{R,K}} Q$ *then* $P^\sharp\equiv Q^\sharp$.
(5) *If* $P\to_{\exists_{R,K}} Q$ *then* $P^\sharp =_{\lambda\mu} Q^\sharp$.
(6) *If* $P\to_{\mathrm{sp}_K} Q$ *then* $P^\sharp\to^*_\eta Q^\sharp$.

Proof. By induction on the derivations together with the previous substitution lemma for the case (2). We show an outline for this case. Let K be $(U_1,\ldots,U_m,\pi_2^n d)$ with $m,n\ge 0$. Now we prove the case P of $\lambda b:B^*.(\lambda a:A^*.R(\pi_2^q c))K$ with $A,B\not\equiv\bot$ and $c:C_1^k\wedge\cdots\wedge C_q^k\wedge C^*_{q+1}$ where $C_{q+1}\not\equiv\bot$.

Case 1. Every U_i $(1\le i\le m)$ is a term:

Assume that $d : D_1^k \wedge \cdots \wedge D_n^k \wedge D_{n+1}^*$ where $D_{n+1} \not\equiv \bot$, and then we have the following to be proved:

$$
\begin{aligned}
&(\lambda b\!:\!B^*.(\lambda a\!:\!A^*.R(\pi_2^q c))K)^\sharp \\
&\quad = \quad \mu b\!:\!B.[d](\lambda d_1\!:\!D_1 \ldots \lambda d_n\!:\!D_n.(\mu a\!:\!A.[c](\lambda c_1\!:\!C_1 \ldots \lambda c_q\!:\!C_q.R^\sharp))U_1^\sharp \ldots U_m^\sharp) \\
&=_{\lambda\mu} (\lambda b\!:\!B^*.R(\pi_2^q c)[a := K])^\sharp
\end{aligned}
$$

The last line above can be verified following the same pattern as given by the proof of Proposition 5 in [3], with case analysis of $a \equiv c$ or not; $q+1 \leq m$ or not; or $b \equiv d$ or not.

Case 2. U_j is a type for some $1 \leq j \leq m$ and every U_i $(i < j)$ is a term:
We can assume that $d : D_1^k \wedge \cdots \wedge D_{j-1}^k \wedge \exists X.D_j^*$ and $n = j-1$. This case can also be confirmed by the similar pattern to the case 1 above.

The case of (5) can be proved by the use of the case (2). □

Theorem 1. **(i)** *Let* $M_1, M_2 \in \Lambda 2\mu$.
$\Gamma \vdash M_1 =_{\lambda\mu} M_2 : A \mid \Delta$ *if and only if* $\Gamma^k, \Delta^* \vdash [\![M_1]\!] =_{\lambda^\exists} [\![M_2]\!] : A^k$.
(ii) *Let* $P_1, P_2 \in Univ_{2\mu}$.
$\Gamma^k, \Delta^* \vdash P_1 =_{\lambda^\exists} P_2$ *if and only if* $\Gamma \vdash P_1^\sharp =_{\lambda\mu} P_2^\sharp : A \mid \Delta$.

Proof. **(i)** From Propositions 1, 3 and 4 and Lemma 6 (1).
(ii) From Propositions 1, 3 and 4 and Lemma 6 (2). □

6 Concluding remarks and related work

We have shown that the CPS-translation is sound and complete for second order $\lambda\mu$-calculus with (η)-rule. In particular, the inverse translation forms a bijection between the universe $Univ_{2\mu}$ and $\lambda 2\mu$, i.e., intuitionistic and classical proofs. In order to prove the completeness, we deeply apply the extensionality (η) of the source calculus. The extensionality at a term-level can be valid in the target calculus with surjective pairing (sp), and $\forall X$-types can be validated with second order weak sums under the duality between $\forall$ and $\exists$ like in [15, 18]. Similarly, extensionality at a type-level: $\lambda X.MX \to M$ if $X \notin FV(M)$ could be valid with strong sums. However, in this case the target calculus becomes impredicative, which leads to a contradiction as proved by Coquand [17]. This is the reason why the target calculus has weak sums rather than strong ones.

Another important observation on the target calculus is that $\lambda^\exists$ can be regarded as a subsystem of the system **F** of Girard, in the sense that $\bot, \wedge, \exists$ can be coded in **F** [4]. A refinement of Proposition 1 (Soundness) means that the strong normalization property of $\lambda^\exists$ implies that of the system **F**, since one step reduction relation of **F** can be interpreted as the transitive closure of one step of $\lambda^\exists$ via the CPS-translation. This remark leads to yet another proof of the strong normalization of **F**[1].

[1] Th. Coquand kindly commented on the target calculus $\lambda^\exists$.

The idea of the novel CPS-translation itself is given by Hofmann-Streicher [6] in the simply typed case, and this is followed by Streicher-Reus [16] and Selinger [15]. The same idea has also appeared in Filinski [1] as a context of a function application, such that a function $f : A \to B$ is viewed as a function closure accepting the pair consisting of an A-typed value (an argument value) and a B-accepting continuation (a rest continuation to the final result). Our contribution in this paper is the definition of the inverse translation for the CPS-translation, which is derived from the CPS-grammar and an analysis on the forms of terms in the universe $Univ_{2\mu}$. From this we have been able to prove the completeness that is the first result, to our knowledge, in the second order typed case.

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