Infinite sum representation algorithm of continued fraction expansions, II

Takao Komatsu
Department of Mathematical System Science
Faculty of Science and Technology
Hirosaki University, Hirosaki, 036-8561, Japan
komatsu@cc.hirosaki-u.ac.jp

MR Subject Classifications: 11A55, 11J70, 11Y16
Key words: Continued fractions, Infinite sums

Abstract

A reverse algorithm of continued fractions toward real numbers via power series was presented at the meeting of JANT in May 2003. This algorithm works well if every partial quotient of the continued fraction is larger than 1. However, it does not always reflect the facts if there are many 1's in the partial quotients. This article shows some methods how to compensate such a defect.

1 Introduction

For any real number α, its continued fraction $\alpha = [a_0; a_1, a_2, a_3, \dots]$ can be uniquely expressed by the algorithm:

$$\begin{aligned} \alpha &= a_0 + (1/\alpha_1), & a_0 &= \lfloor \alpha \rfloor, \\ \alpha_n &= a_n + (1/\alpha_{n+1}), & a_n &= \lfloor \alpha_n \rfloor \quad (n = 1, 2, \dots). \end{aligned}$$

Namely,

$$[a_0; a_1, a_2, a_3, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

However, on the contrary, for any given sequence of partial quotients $a_0, a_1, a_2, a_3, \ldots$, is there any general algorithm to find the real number yielding the continued fraction expansion $[a_0; a_1, a_2, a_3, \ldots]$? If the partial quotients is finite, it is not difficult to find its corresponding rational number. If it is infinite and periodic, it is still possible to find its corresponding quadratic irrational number. But in the other cases it is too hard to find an explicit or recognizable form of its corresponding real number from the continued fraction.

The following reverse algorithm of continued fractions toward real numbers via power series was presented at the 10th meeting of JANT in May 2003.

Theorem A Let $Q(k)$ be a function in k which takes positive integral values for $k = 1, 2, \ldots$. Then

$$[0; \overline{Q(k)}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{1}{Q(1)} \sum_{n=1}^{\infty} (-1)^{n-1} s_{n,1}\,,$$

where for $n = 0, 1, 2, \ldots$

$$c'_n = \frac{1}{Q(1)} (c_n - c_{n-1} s_{1,1} + c_{n-2} s_{2,1} - \cdots + (-1)^{n-1} c_1 s_{n-1,1} + (-1)^n c_0 s_{n,1})$$

and for $k \geq 1$

$$s_{n,k} = \frac{1}{Q(k)Q(k+1)} (s_{n-1,1} + s_{n-1,2} + \cdots + s_{n-1,k+1}) \quad (n \geq 2)$$

with

$$s_{1,k} = \frac{1}{Q(k)Q(k+1)} \quad \text{and} \quad s_{0,k} = 1\,.$$

One of the immediate applicable forms is as follows.

Corollary B Let $Q(k)$ be a function in k which takes positive integral values for $k = 1, 2, \ldots$. Then

$$[0; \overline{Q(k)}]_{k=1}^{\infty} = \frac{\frac{1}{Q(1)} \left(1 + \frac{1}{Q(2)Q(3)} \sum_{n=1}^{\infty} (-1)^{n-1} d_{n,1}\right)}{1 + \frac{1}{Q(1)Q(2)} + \frac{1}{Q(2)Q(3)} \sum_{n=1}^{\infty} (-1)^{n-1} d_{n,1}}\,,$$

where for $k \geq 1$

$$d_{n,k} = \frac{1}{Q(k+2)Q(k+3)}(d_{n-1,1} + d_{n-1,2} + \cdots + d_{n-1,k+1}) \quad (n \geq 3)$$

with

$$d_{2,k} = \frac{1}{Q(k+2)Q(k+3)} \quad \text{and} \quad d_{1,1} = 1\,.$$

This algorithm is valid for any pattern of continued fraction expansion as long as the sum above converges. For example, if $a_k \geq 2$ for all $k \geq 1$, then the series in Theorem A and Corollary B converges.

Example 1 Let a be a positive integer, u be rational so that ua is a positive integer. Then

$$[0; \overline{ua^k}]_{k=1}^{\infty} = \frac{\sum_{s=0}^{\infty} u^{-2s-1} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} u^{-2s} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}\,.$$

Example 2

$$[0; 1^2, 2^2, 3^2, 4^2, \dots] = [0; \overline{k^2}]_{k=1}^{\infty}$$
$$= \frac{1 + \frac{1}{36}\left(1 - \frac{1}{144} + \frac{17}{259200} - \frac{149}{233280000} + \frac{257063}{41150592000000} - \cdots\right)}{1 + \frac{1}{4} + \frac{1}{36}\left(1 - \frac{1}{144} + \frac{17}{259200} - \frac{149}{233280000} + \frac{257063}{41150592000000} - \cdots\right)}\,.$$

$$[0; 1^2, 2^2, 3^2, 4^2, \dots]$$
$$= 1 - \frac{1}{4} + \frac{5}{72} - \frac{401}{20736} + \frac{201017}{37324800} - \frac{100767923}{67184640000} + \frac{2475186587813}{5925685248000000} - \cdots\,.$$

Example 3

$$[0; q^{\binom{0}{2}}, q^{\binom{1}{2}}, q^{\binom{2}{2}}, q^{\binom{3}{2}}, q^{\binom{4}{2}}, \dots] = [0; \overline{q^{\binom{k-1}{2}}}]_{k=1}^{\infty}$$
$$= \frac{1 + q^{-1} - q^{-5} + q^{-5}(q^{-4} + q^{-9}) - q^{-5}(q^{-4}(q^{-4} + q^{-9}) + q^{-9}(q^{-4} + q^{-9} + q^{-16})) + \cdots}{2 + q^{-1} - q^{-5} + q^{-5}(q^{-4} + q^{-9}) - q^{-5}(q^{-4}(q^{-4} + q^{-9}) + q^{-9}(q^{-4} + q^{-9} + q^{-16})) + \cdots}\,,$$

where $\binom{n}{2}$ are binomial coefficients with $\binom{0}{2} = \binom{1}{2} = 0$.

Example 4

$$[0; F_0, F_1, F_2, F_3, \dots] = [0; 1, 1, 2, 3, 5, 8, 13, \dots]$$
$$= \frac{1 + \frac{1}{2}\left(1 - \frac{1}{6} + \frac{7}{180} - \frac{101}{10800} + \frac{19009}{8424000} - \frac{25053317}{45995040000} + \frac{561357912257}{4269259612800000} - \cdots\right)}{2 + \frac{1}{2}\left(1 - \frac{1}{6} + \frac{7}{180} - \frac{101}{10800} + \frac{19009}{8424000} - \frac{25053317}{45995040000} + \frac{561357912257}{4269259612800000} - \cdots\right)},$$

where $\{F_n\}$ are Fibonacci numbers defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ $(n \ge 2)$.

2 Does this algorithm really retrieve the original real number?

For instance, we shall consider a simple case. Let $Q(k) = a$ $(k = 1, 2, \dots)$ in Corollary B, where a is an integer with $a > 1$. Then

$$d_{2,k} = \frac{1}{a^2}, \quad d_{3,k} = \frac{k+1}{a^3}, \quad d_{4,k} = \frac{(k+1)(k+4)}{2a^6}, \quad \dots$$

In general, we have

$$d_{n,k} = \frac{(k+1)(2n+k-4)!}{(n-2)!(n+k-1)!a^{2n-2}}.$$

Hence,

$$d_{n,1} = \frac{2(2n-3)!}{(n-2)!n!a^{2n-2}}.$$

Since

$$\sqrt{1 + \frac{4}{a^2}} = 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2} - 1) \cdots (\frac{1}{2} - (n-1))}{n!} \left(\frac{4}{a^2}\right)^n$$
$$= 1 + \frac{2}{a^2} + \frac{4}{a^2} \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{(n-1)!(n+1)!a^{2n}},$$

we have

$$[0;a,a,\dots]=\frac{\dfrac{1}{a}\left(1+\dfrac{1}{a^2}\left(1+\sum_{n=2}^\infty\dfrac{2(-1)^{n-1}(2n-3)!}{(n-2)!n!a^{2n-2}}\right)\right)}{1+\dfrac{1}{a^2}+\dfrac{1}{a^2}\left(1+\sum_{n=2}^\infty\dfrac{2(-1)^{n-1}(2n-3)!}{(n-2)!n!a^{2n-2}}\right)}$$

$$=\frac{\dfrac{1}{a}\left(1+\dfrac{1}{a^2}\left(1+2\sum_{n=1}^\infty\dfrac{(-1)^{n}(2n-1)!}{(n-1)!(n+1)!a^{2n}}\right)\right)}{1+\dfrac{1}{a^2}+\dfrac{1}{a^2}\left(1+2\sum_{n=1}^\infty\dfrac{(-1)^{n}(2n-1)!}{(n-1)!(n+1)!a^{2n}}\right)}$$

$$=\frac{\dfrac{1}{a}\left(1+\dfrac{\sqrt{a^2+4}-a}{2a}\right)}{1+\dfrac{1}{a^2}+\dfrac{\sqrt{a^2+4}-a}{2a}}=\frac{\sqrt{a^2+4}-a}{2},$$

as expected.

However, if $a=1$, then the power series $\sum_{n=1}(-1)^{n-1}d_{n,1}$ does not converge. See Example 8 below.

Consider the case where $\alpha=[0;1,1,1,2,1,3,1,4,1,5,\dots]$. If we apply Corollary B above, then we calculate $d_{2,1}=1/2$, $d_{3,1}=1/2$, $d_{4,1}=7/12$, $d_{5,1}=13/18$, $d_{6,1}=397/432$, $d_{7,1}=3073/2592$, $d_{8,1}=119807/77760$, $d_{9,1}=4689709/2332800$, $d_{10,1}=183987383/69984000$, $\dots$. But, this yields the unexpected simple continued fraction $[0;5,15,1,2,1,27,\dots]$ instead of the desired one $[0;1,1,1,2,1,3,1,4,1,5,\dots]$.

Hence, if 1 appears so often in the partial quotients, Theorem A with Corollary B cannot be applied directly as it is.

3 Main result

We shall describe a more general algorithm which can be applied to the case where 1 appears very often and regularly in the sequence of partial quotients. By using Theorem 1 below we properly have the desired simple continued fraction.

Let $\epsilon_k=1$ or -1 ($k=2,3,\dots$), and $S(k)=\epsilon_{k+1}/(a_k a_{k+1})$ ($k=1,2,\dots$).

Theorem 1

$$\cfrac{1}{a_1+\cfrac{\epsilon_2}{a_2+\cfrac{\epsilon_3}{a_3+\dots}}}=\frac{\sum_{n=0}^\infty c'_n}{\sum_{n=0}^\infty c_n}=\frac{1}{a_1}\sum_{n=0}^\infty(-1)^{n-1}s_{n,1}\,,$$

where for $n=0,1,2,\dots$

$$c'_n=\frac{1}{a_1}(c_n-c_{n-1}s_{1,1}+c_{n-2}s_{2,1}-\dots+(-1)^{n-1}c_1s_{n-1,1}+(-1)^n c_0 s_{n,1})$$

and for $k\ge 1$

$$s_{n,k}=(s_{n-1,1}+s_{n-1,2}+\dots+s_{n-1,k+1})S(k)\quad(n\ge 2)$$

with $s_{1,k}=S(k)$ and $s_{0,k}=1$.

One of the immediate applicable forms is as follows.

Corollary 1

$$\cfrac{1}{a_1+\cfrac{\epsilon_2}{a_2+\cfrac{\epsilon_3}{a_3+\dots}}}=\frac{1+S(2)\sum_{n=1}^\infty(-1)^{n-1}d_{n,1}}{a_1\bigl(1+S(1)+S(2)\sum_{n=1}^\infty(-1)^{n-1}d_{n,1}\bigr)}=\frac{1}{a_1}\sum_{n=1}^\infty(-1)^{n-1}s_{n,1}\,,$$

where for $k\ge 1$

$$d_{n,k}=(d_{n-1,1}+d_{n-1,2}+\dots+d_{n-1,k+1})S(k+2)\quad(n\ge 3)$$

with $d_{2,k}=S(k+2)$ and $d_{1,1}=1$.

Furthermore,

$$\cfrac{1}{a_3+\cfrac{\epsilon_4}{a_4+\cfrac{\epsilon_5}{a_5+\dots}}}=\frac{1}{a_3}\sum_{n=1}^\infty(-1)^{n-1}d_{n,1}\,.$$

If $\epsilon_2=\epsilon_3=\epsilon_4=\dots=1$, then this is just Theorem A with Corollary B.

4 Application I (Negative continued fractions)

If $\epsilon_2 = \epsilon_3 = \epsilon_4 = \cdots = -1$, then this is called the *negative* continued fraction expansion. The negative continued fraction expansion of a real α with $0 < \alpha < 1$ is denoted by

$$\alpha = {}^{-}[0; a_1, a_2, a_3, \dots] = \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 - \dots}}}$$

where the integers $a_i \ge 2$ are generated by ceiling functions rather than floor functions in the continued fraction algorithm:

$$\begin{aligned} \theta_0 &= \{\alpha\}, \\ 1/\theta_{n-1} &= a_n - \theta_n, \qquad a_n = \lceil 1/\theta_{n-1} \rceil \qquad (n \ge 1). \end{aligned}$$

with corresponding convergents $p_n/q_n = {}^{-}[0; a_1, \dots, a_n]$ given by

$$\begin{aligned} p_n &= a_n p_{n-1} - p_{n-2} \quad (n \ge 1), \quad p_0 = 0, \quad p_{-1} = -1, \\ q_n &= a_n q_{n-1} - q_{n-2} \quad (n \ge 1), \quad q_0 = 1, \quad q_{-1} = 0. \end{aligned}$$

A simple continued fraction expansion can be transformed into a negative continued fraction expansion by the following rule;

$${}^{-}[0; a_1, a_2, a_3, \dots] = [0; a_1 - 1, 1, a_2 - 2, 1, a_3 - 2, 1, a_4 - 2, 1, \dots].$$

In this continued fractions 1 appears every second time in the sequence of partial quotients.

When $[0; a_1, a_2, a_3, \dots]$ corresponds to the tanh-type continued fraction, ${}^{-}[0; a_1, a_2, a_3, \dots]$ corresponds to the tan-type continued fraction. In this special case, Theorem 1 can be reduced as follows.

Theorem 2 If $a_1 > 1$ and $a_k > 2$ $(k \ge 2)$, then

$${}^{-}[0; a_1, a_2, \dots] = [0; a_1 - 1, \overline{1, a_k - 2}]_{k=2}^{\infty} = \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{1}{a_1} \sum_{n=0}^{\infty} s_{n,1} \,,$$

where for $n = 0, 1, 2, \dots$

$$c'_n = \frac{1}{a_1}(c_n + c_{n-1}s_{1,1} + c_{n-2}s_{2,1} + \cdots + c_1 s_{n-1,1} + c_0 s_{n,1})$$

and for $k \ge 1$

$$s_{n,k} = \frac{1}{a_k a_{k+1}} \sum_{i=1}^{k+1} s_{n-1,i} \quad (n \ge 2)$$

with

$$s_{1,k} = \frac{1}{a_k a_{k+1}} \quad \text{and} \quad s_{0,k} = 1\,.$$

Corollary 2 If $a_1 > 1$ and $a_k > 2$ $(k \ge 2)$, then

$$\begin{aligned} {}^{-}[0; a_1, a_2, \dots] &= [0; a_1 - 1, \overline{1, a_k - 2}]_{k=2}^{\infty} \\ &= \frac{\frac{1}{a_1}\left(1 - \frac{1}{a_2 a_3}(1 + \sum_{n=2}^{\infty} d_{n,1})\right)}{1 - \frac{1}{a_1 a_2} - \frac{1}{a_2 a_3}(1 + \sum_{n=2}^{\infty} d_{n,1})}\,, \end{aligned}$$

where for $k \ge 1$

$$d_{n,k} = \frac{1}{a_{k+2} a_{k+3}} \sum_{i=1}^{k+1} d_{n-1,i} \quad (n \ge 3)$$

with

$$d_{2,k} = \frac{1}{a_{k+2} a_{k+3}}\,.$$

Furthermore,

$${}^{-}[0; a_3, a_4, \dots] = [0; a_3 - 1, \overline{1, a_k - 2}]_{k=4}^{\infty} = \frac{1}{a_3} \sum_{n=1}^{\infty} d_{n,1}\,.$$

Example 5 Set $a_k = k + 1$ in Theorem 2. From $d_{2,1} = 1/20$, $d_{3,1} = 1/240$, $d_{4,1} = 13/33600$, $d_{5,1} = 1/26880$, $d_{6,1} = 263/72576000$, $d_{7,1} = 103/290304000$, $d_{8,1} = 15563/447068160000$, $d_{9,1} = 55019/16094453760000$, $d_{10,1} = 127753/380414361600000$, $\dots$, we have

$$\begin{aligned} &[0; 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, \dots] = [0; \overline{1, k}]_{k=1}^{\infty} \\ &= \frac{2\left(1 - \frac{1}{6} - \frac{1}{12}(1 + \frac{1}{20} + \frac{1}{240} + \frac{13}{33600} + \frac{1}{26880} + \frac{263}{72576000} + \frac{103}{290304000} + \cdots)\right)}{1 - \frac{1}{12}(1 + \frac{1}{20} + \frac{1}{240} + \frac{13}{33600} + \frac{1}{26880} + \frac{263}{72576000} + \frac{103}{290304000} + \cdots)} - 1 \\ &= \frac{1 - \frac{1}{3} - \frac{1}{12}(1 + \frac{1}{20} + \frac{1}{240} + \frac{13}{33600} + \frac{1}{26880} + \frac{263}{72576000} + \frac{103}{290304000} + \cdots)}{1 - \frac{1}{12}(1 + \frac{1}{20} + \frac{1}{240} + \frac{13}{33600} + \frac{1}{26880} + \frac{263}{72576000} + \frac{103}{290304000} + \cdots)}\,. \end{aligned}$$

Example 6 Set $a_k = (2k-1)a$ in Theorem 2, where a is an integer with $a > 1$. When

$$c_k = \frac{(-1)^k}{(2k)!a^{2k}} \quad (k = 0, 1, 2, \dots, n)\,,$$

we can get

$$c'_n = \frac{(-1)^n}{(2n+1)!a^{2n+1}}\,.$$

Thus,

$$\begin{aligned}
{}^{-}[0; \overline{(2k-1)a}]_{k=1}^{\infty} &= [0; a-1, \overline{1, (2k+1)a-2}]_{k=1}^{\infty} \\
&= \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!a^{2n+1}}}{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!a^{2n}}} = \frac{\sin\frac{1}{a}}{\cos\frac{1}{a}} = \tan\frac{1}{a}\,.
\end{aligned}$$

5 Application II

If $\epsilon_2 = \epsilon_4 = \cdots = -1$ and $\epsilon_3 = \epsilon_5 = \cdots = 1$, the general continued fraction is transformed to the following simple continued fraction;

$$\begin{aligned}
&\frac{1}{a'_1 -}\,\frac{1}{a'_2 +}\,\frac{1}{a'_3 -}\,\frac{1}{a'_4 +}\,\frac{1}{a'_5 -}\cdots \\
&= [0; a'_1, -a'_2, -a'_3, a'_4, a'_5, -a'_6, -a'_7, \dots] \\
&= [0; a'_1 - 1, 1, a'_2 - 1, a'_3, -a'_4, -a'_5, a'_6, a'_7, \dots] \\
&= [0; a'_1 - 1, 1, a'_2 - 1, a'_3 - 1, 1, a'_4 - 1, 1, a'_5, -a'_6, -a'_7, \dots] = \dots \\
&= [0; a'_1 - 1, 1, a'_2 - 1, a'_3 - 1, 1, a'_4 - 1, 1, a'_5 - 1, 1, a'_6 - 1, \dots]\,.
\end{aligned}$$

In this case, Theorem 1 with Corollary 1 can be reduced as follows.

Theorem 3 If $a_k > 1$ $(k \ge 1)$, then

$$[0; \overline{a_{2k-1} - 1, 1, a_{2k} - 1}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{1}{a_1}\sum_{n=0}^{\infty}(-1)^{n-1} s_{n,1}\,,$$

where for $n = 0, 1, 2, \dots$

$$c'_n = \frac{1}{a_1}(c_n - c_{n-1}s_{1,1} + c_{n-2}s_{2,1} - \cdots + (-1)^{n-1}c_1 s_{n-1,1} + (-1)^n c_0 s_{n,1})$$

and for $k \geq 1$

$$s_{n,k} = \frac{(-1)^k}{a_k a_{k+1}} \sum_{i=1}^{k+1} s_{n-1,i} \quad (n \geq 2)$$

with

$$s_{1,k} = \frac{(-1)^k}{a_k a_{k+1}} \quad \text{and} \quad s_{0,k} = 1\,.$$

Corollary 3 If $a_k > 1$ $(k \geq 1)$, then

$$[0; \overline{a_{2k-1} - 1, 1, a_{2k} - 1}]_{k=1}^{\infty} = \frac{\frac{1}{a_1}\left(1 + \frac{1}{a_2 a_3}\sum_{n=1}^{\infty}(-1)^{n-1} d_{n,1}\right)}{1 - \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3}\sum_{n=1}^{\infty}(-1)^{n-1} d_{n,1}}\,,$$

where for $k \geq 1$

$$d_{n,k} = \frac{(-1)^k}{a_{k+2} a_{k+3}} \sum_{i=1}^{k+1} d_{n-1,i} \quad (n \geq 3)$$

with

$$d_{2,k} = \frac{(-1)^k}{a_{k+2} a_{k+3}} \quad \text{and} \quad d_{1,1} = 1\,.$$

Furthermore,

$$[0; \overline{a_{2k-1} - 1, 1, a_{2k} - 1}]_{k=2}^{\infty} = \frac{1}{a_3} \sum_{n=1}^{\infty} (-1)^{n-1} d_{n,1}\,.$$

Example 7 Put $a_{2k-1} = (2k-1)s$ $(s \geq 2)$ and $a_{2k} = 2$ $(k = 1, 2, \dots)$ in Corollary 3. Then by

$$d_{2,1} = -\frac{1}{6s} \quad d_{3,1} = \frac{1}{90s^2} \quad d_{4,1} = \frac{1}{1080s^3} \quad d_{5,1} = -\frac{1}{4536s^4} \quad \dots$$

we have

$$\begin{aligned} e^{1/s} - 1 &= [0; \overline{(2k-1)s - 1, 1, 1}]_{k=1}^{\infty} \\ &= \frac{\frac{1}{s}\left(1 + \frac{1}{6s}\left(1 + \frac{1}{6s} + \frac{1}{90s^2} - \frac{1}{1080s^3} - \frac{1}{4536s^4} + \cdots\right)\right)}{1 - \frac{1}{2s} + \frac{1}{6s}\left(1 + \frac{1}{6s} + \frac{1}{90s^2} - \frac{1}{1080s^3} - \frac{1}{4536s^4} + \cdots\right)}\,. \end{aligned}$$

This can also be applied to $e - 1 = [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots]$, simply setting $s = 1$.

Example 8 Put $a_k = 2$ $(k = 1, 2, \dots)$ in Corollary 3. Then we note $d_{2,1} = -1/4$, $d_{4,1} = 1/64$, $d_{6,1} = -1/512$, $d_{8,1} = 5/16384$, $d_{10,1} = -7/131072$, $d_{12,1} = 21/2097152$, $d_{14,1} = -33/16777216$, $d_{16,1} = 429/1073741824$, $\dots$ and $d_{3,1} = d_{5,1} = d_{7,1} = \cdots = 0$, and so we have

$$[0; 1, 1, 1, \dots] = \frac{\frac{1}{2}\left(1 + \frac{1}{4}\left(1 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \frac{7}{131072} - \frac{21}{2097152} + \dots\right)\right)}{1 - \frac{1}{4} + \frac{1}{4}\left(1 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \frac{7}{131072} - \frac{21}{2097152} + \dots\right)}.$$

The right-hand side should be equal to $(\sqrt{5}-1)/2$. Indeed, by induction we have for $n = 1, 2, \dots$ and $k = 1, 2, \dots$

$$d_{2n,2k-1} = \frac{(-1)^n k(k+2n-3)!}{(n-1)!(k+n-1)!4^{2n-1}}, \qquad d_{2n,2k} = \frac{(-1)^{n+1} k(k+2n-3)!}{(n-1)!(k+n-1)!4^{2n-1}},$$

$$d_{2n+1,2k-1} = 0, \qquad d_{2n+1,2k} = \frac{(-1)^n (k+1)(k+2n-2)!}{(n-1)!(k+n)!4^{2n}}.$$

Since

$$\begin{aligned}\sqrt{1 + \frac{4}{a^2}} &= 1 + \sum_{n=1}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-(n-1))}{n!}\left(\frac{4}{a^2}\right)^n \\ &= 1 + \frac{2}{a^2} + \frac{4}{a^2}\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)!}{(n-1)!(n+1)!a^{2n}},\end{aligned}$$

we have

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1} d_{n,1} &= 1 - \sum_{n=1}^{\infty} \frac{(-1)^n(2n-2)!}{(n-1)!n!4^{2n-1}} \\ &= 1 + \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n(2n-1)!}{(n-1)!(n+1)!4^{2n}} = \sqrt{5} - 1,\end{aligned}$$

yielding

$$\begin{aligned}\frac{\frac{1}{2}\left(1 + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}d_{n,1}\right)}{1 - \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^{n-1}d_{n,1}} &= \frac{\frac{1}{2}\left(1 + \frac{1}{4}(\sqrt{5}-1)\right)}{1 - \frac{1}{4} + \frac{1}{4}(\sqrt{5}-1)} \\ &= \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \dots].\end{aligned}$$

6 Application III

Put

$$\epsilon_k = \begin{cases} -1, & \text{if } k \equiv 2 \pmod 4; \\ 1, & \text{otherwise.} \end{cases}$$

Then

$$\frac{1}{a'_1} \frac{1}{-a'_2} \frac{1}{+a'_3} \frac{1}{+a'_4} \frac{1}{+a'_5} \frac{1}{-a'_6} \frac{1}{+a'_7} \frac{1}{+a'_8} \frac{1}{+a'_9} \frac{1}{-a'_{10}} \frac{1}{+a'_{11}} \frac{1}{+a'_{12}} \frac{}{+} \cdots$$
$$= [0; a'_1, -a'_2, -a'_3, -a'_4, -a'_5, a'_6, a'_7, a'_8, a'_9, -a'_{10}, -a'_{11}, -a'_{12}, -a'_{13}, a'_{14}, \dots]$$
$$= [0; a'_1 - 1, 1, a'_2 - 1, a'_3, a'_4, a'_5 - 1, 1, a'_6 - 1, a'_7, a'_8, a'_9 - 1, 1, a'_{10} - 1, \dots] .$$

In this case, Theorem 1 with Corollary 1 can be reduced as follows.

Theorem 4 If $a_{4k-3} > 1$ and $a_{4k-2} > 1$ $(k \ge 1)$, then

$$[0; \overline{a_{4k-3} - 1, 1, a_{4k-2} - 1, a_{4k-1}, a_{4k}}]_{k=1}^{\infty} = \frac{\sum_{n=0}^{\infty} c'_n}{\sum_{n=0}^{\infty} c_n} = \frac{1}{a_1} \sum_{n=0}^{\infty} (-1)^{n-1} s_{n,1} ,$$

where for $n = 0, 1, 2, \dots$

$$c'_n = \frac{1}{a_1}(c_n - c_{n-1}s_{1,1} + c_{n-2}s_{2,1} - \cdots + (-1)^{n-1}c_1 s_{n-1,1} + (-1)^n c_0 s_{n,1})$$

and for $k \ge 1$

$$s_{n,k} = \frac{\epsilon_{k+1}}{a_k a_{k+1}} \sum_{i=1}^{k+1} s_{n-1,i} \quad (n \ge 2)$$

with

$$s_{1,k} = \frac{\epsilon_{k+1}}{a_k a_{k+1}} \quad \text{and} \quad s_{0,k} = 1 .$$

Corollary 4 If $a_{4k-3} > 1$ and $a_{4k-2} > 1$ $(k \ge 1)$, then

$$[0; \overline{a_{4k-3} - 1, 1, a_{4k-2} - 1, a_{4k-1}, a_{4k}}]_{k=1}^{\infty} = \frac{\frac{1}{a_1}\left(1 + \frac{1}{a_2 a_3} \sum_{n=1}^{\infty} (-1)^{n-1} d_{n,1}\right)}{1 - \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} \sum_{n=1}^{\infty} (-1)^{n-1} d_{n,1}} ,$$

where for $k \ge 1$

$$d_{n,k} = \frac{\epsilon_{k+1}}{a_{k+2} a_{k+3}} \sum_{i=1}^{k+1} d_{n-1,i} \quad (n \ge 3)$$

with

$$d_{2,k}=\frac{\epsilon_{k+1}}{a_{k+2}a_{k+3}}\quad\text{and}\quad d_{1,1}=1\,.$$

Furthermore,

$$[0;\overline{a_{4k-1},a_{4k},a_{4k+1}-1,1,a_{4k+2}-1}]_{k=1}^\infty=\frac{1}{a_3}\sum_{n=1}^\infty(-1)^{n-1}d_{n,1}\,.$$

Example 9 Put $a_{4k-3}=a_{4k-1}=3k$, $a_{4k-2}=2$ and $a_{4k}=12k+6$ ($k=1,2,\dots$) in Corollary 4. Then this continued fraction equals

$$\begin{aligned}[0;\overline{3k-1,1,1,3k,12k+6}]_{k=1}^\infty&=\frac{\frac{1}{3}\left(1+\frac{1}{6}(1-\frac{1}{54}+\frac{1}{1944}-\cdots)\right)}{1-\frac{1}{6}+\frac{1}{6}(1-\frac{1}{54}+\frac{1}{1944}-\cdots)}\\&=0.38905609\cdots=e^2-7\,.\end{aligned}$$

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