DENOTATIONAL SEMANTICS EXCLUDING WEAK-EXTENSIONALITY IN SIMPLE TYPES

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1. Introduction

In the study of denotational semantics of programming languages, we often employ the notion of function as a denotation of a program. Actually, for a wide variety of systems of λ-calculus, a number of successful mathematical frameworks have been obtained so far according to this line. For example, the models of type-free λ-calculus presented in [3, Chapter 18] and [10] are well-known under the name of λ-*model* nowadays. However, on the other hand it seems rather strong to ignore the internal feature of algorithms by means of the extensionality of functions. In this explication, we considering two algorithms with different internal structures, their denotations come out to be identified when they always return the same result of application.

Contrary to this strong aspect of the ordinary semantics, we make an attempt to present a general framework of semantics in which β-equality of λ-calculus is ensured to be sound without using the notion of extensionality. In terms of semantics of type-free λ-calculus, such structures exactly correspond to the notion of λ-*algebras*. For this requirement, we need another mathematical notion to model λ-abstraction and application in particular, for which we adopt the arrows of a version of free semi-cartesian closed category and introduce a notion of their application. This induces two viewpoint of interpretation. One is a certain fine viewpoint to capture an internal structure of λ-abstraction, and the other is a coarse viewpoint to evaluate the result of application.

As a preliminary study on the motivation above, we first confine our attention to the syntax of simply typed λ-calculus. Hence, we need not ensure the existence of any fixed point of the arrows and any isomorphism among the objects. This means that we do not employ any result of domain theory for our construction. As a future work, we leave the problem to incorporate domain theoretical discussion into our construction, which might

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lead us to models working on stronger paradigms of computation, such as PCF and type-free λ-calculus.

The content of this paper is as follows: In Section 2, we review the syntax of simply typed λ-calculus and its models. We especially take two notions of models under our consideration by analogy with the semantical study of type-free λ-calculus. One is a counterpart of the notion of λ-algebra in which we only require that denotations of λ-terms are invariant under β-equality. The other is a counterpart of the notion of λ-model, which is endowed with the property of weak-entensionality. In Section 3, we are to present a free-category $\mathscr{F}_C$ so as to accomplish our expecting semantics. The construction of $\mathscr{F}_C$ is similar to the one studied in [8, Chapter 1], and we first introduce an underlying graph $\mathscr{F}_G$. In the process of generating $\mathscr{F}_G$, we explicitly define a notion of application of arrows by means of a reduction for evaluation. We then introduce an equality on arrows. It is substantially the equality of semi-cartesian closed category, but comes out to be comparable with the equality of cartesian closed category for arrows applied to a member of their domain. This presentation of equality would be a key to manipulate the information concerning extensionality of models. In Section 4, we finally present our model $\mathscr{B}$ of simply typed λ-calculus. As for the model $\mathscr{B}$, we know that η-axiom is sound but η-equality is not in general, which inevitably entails the fact that weak-extensionality does not hold in $\mathscr{B}$. This sharply contrast to the model obtained from $\mathscr{F}_C$ by following the standard construction in [5, Chapter 3], in which it is not clear whether weak-extensionality holds or not.

2. Simply typed λ-calculus and its models

We fix a set of atomic types throughout this paper, from which the set of *simple types* is generated by the following abstract syntax:

$$\sigma ::= \alpha \mid 1 \mid \sigma \times \sigma \mid \sigma \to \sigma$$

in which α varies over the set of type constants. We restrict our attention only to typed λ-terms of these simple types which are inductively generated by the following rules:

$$\text{(Var)} \quad x^\sigma : \sigma \qquad\qquad \text{(Fst)} \quad \frac{M : \sigma \times \tau}{\mathrm{fst}(M) : \sigma}$$

$$\text{(Pair)} \quad \frac{M : \sigma \quad N : \tau}{(M, N) : \sigma \times \tau} \qquad\qquad \text{(Snd)} \quad \frac{M : \sigma \times \tau}{\mathrm{snd}(M) : \tau}$$

$$\text{(App)} \quad \frac{M : \sigma \to \tau \quad N : \sigma}{MN : \tau} \qquad\qquad \text{(Abs)} \quad \frac{M : \tau}{\lambda x^\sigma . M : \sigma \to \tau}$$

We use letters $\sigma, \tau, \upsilon, \ldots$ as meta-variables to designate simple types and $M, N, \ldots$ to designate typed λ-terms, and specify the unique type σ of a λ-term M by the expression $M : \sigma$. For a λ-term M, we write $\mathrm{FV}(M)$ for the set of free-variables appearing in M. For detailed explanation syntax of this system, see [7, Chapter 5] for example.

Next we briefly review some semantical frameworks for simply typed λ-calculus. A *typed applicative structure* is defined by a 5-tuple

$$\langle [\![\,]\!]^{\text{type}}, \mathrm{Fst}, \mathrm{Snd}, \mathrm{Pair}, \mathrm{App}\rangle$$

of functions such that the first component, called *type-interpretation*, assigns a non-empty set $[\![\sigma]\!]^{\text{type}}$ to each simple type σ, and the others assign functions

$$\begin{aligned}
\mathrm{Fst}^{\sigma,\tau} &: [\![\sigma \times \tau]\!]^{\text{type}} \longrightarrow [\![\sigma]\!]^{\text{type}}, \\
\mathrm{Snd}^{\sigma,\tau} &: [\![\sigma \times \tau]\!]^{\text{type}} \longrightarrow [\![\tau]\!]^{\text{type}}, \\
\mathrm{Pair}^{\sigma,\tau} &: [\![\sigma]\!]^{\text{type}} \times [\![\tau]\!]^{\text{type}} \longrightarrow [\![\sigma \times \tau]\!]^{\text{type}}, \\
\mathrm{App}^{\sigma,\tau} &: [\![\sigma \to \tau]\!]^{\text{type}} \times [\![\sigma]\!]^{\text{type}} \longrightarrow [\![\tau]\!]^{\text{type}},
\end{aligned}$$

to each pair of simple types σ and τ, respectively. In a typed applicative structure, an interpretation of free-variables is presented by a mapping ξ, called an *environment*, which maps each term-variable x^σ to an element of $[\![\sigma]\!]^{\text{type}}$. We say that a typed applicative structure is a *weak-extensional model* of simply typed λ-calculus if we are able to determine a meaning of each λ-term with respect to an environment, more precisely, to introduce a mapping $[\![\,]\!]^{\text{term}}$, called *term-interpretation*, which assigns a member $[\![M]\!]^{\text{term}}_\xi$ of $[\![\sigma]\!]^{\text{type}}$ to each pair of environment ξ and λ-term M of type σ, and which satisfies

(1) $\quad \forall x^\sigma \in \mathrm{FV}(M) \quad \xi(x^\sigma) = \rho(x^\sigma) \Longrightarrow [\![M]\!]^{\text{term}}_\xi = [\![M]\!]^{\text{term}}_\rho,$

(2) $\quad [\![x^\sigma]\!]^{\text{term}}_\xi = \xi(x^\sigma),$

(3) $\quad [\![\mathrm{fst}(M)]\!]^{\text{term}}_\xi = \mathrm{Fst}^{\sigma,\tau}([\![M]\!]^{\text{term}}_\xi),$

(4) $\quad [\![\mathrm{snd}(M)]\!]^{\text{term}}_\xi = \mathrm{Snd}^{\sigma,\tau}([\![M]\!]^{\text{term}}_\xi),$

(5) $\quad [\![(M,N)]\!]^{\text{term}}_\xi = \mathrm{Pair}^{\sigma,\tau}([\![M]\!]^{\text{term}}_\xi, [\![N]\!]^{\text{term}}_\xi),$

(6) $\quad [\![MN]\!]^{\text{term}}_\xi = \mathrm{App}^{\sigma,\tau}([\![M]\!]^{\text{term}}_\xi, [\![N]\!]^{\text{term}}_\xi),$

(7) $\quad \mathrm{App}^{\sigma,\tau}([\![\lambda x^\sigma.M]\!]^{\text{term}}_\xi, d) = [\![M]\!]^{\text{term}}_{\xi(x^\sigma : d)},$

(8) $\quad \forall d \in [\![\sigma]\!]^{\text{type}}\ \mathrm{App}^{\sigma,\tau}([\![\lambda x^\sigma.M]\!]^{\text{term}}_\xi, d) = \mathrm{App}^{\sigma,\tau}([\![\lambda x^\sigma.N]\!]^{\text{term}}_\xi, d)$
$\qquad\qquad \Longrightarrow [\![\lambda x^\sigma.M]\!]^{\text{term}}_\xi = [\![\lambda x^\sigma.N]\!]^{\text{term}}_\xi,$

if all types appearing in these expressions are assigned consistently. Here the expression $\xi(x^\sigma : d)$ in (7) designates the environment ξ with the value of the variable x^σ updated to $d \in [\![\sigma]\!]^{\text{type}}$; that is, the value of $\xi(x^\sigma : d)(y^\tau)$ is defined by d if $y^\tau \equiv x^\sigma$, and by $\xi(y^\tau)$ otherwise. We call (8) the property of weak-extensionality. In what follows, we omit the superscripts to distinguish type-interpretation and term-interpretation because of less possibility of confusion, denoting both of them simply by $[\![\,]\!]$.

When a weak-extensional model satisfies even the following strong version of extensionality, it is said to be an *extensional model*.

$$(9) \qquad \forall d \in [\![\sigma]\!] \quad \mathrm{App}^{\sigma,\tau}(f,d) = \mathrm{App}^{\sigma,\tau}(g,d) \implies f = g$$

It is clear that (9) implies (8). In some standard literatures, such as [5] and [9], semantics of simple types have been studied only through the structures comparable with extensional models, possibly under the name of *type-frame* or *Henkin-model*.

One of the reason why we require (8) or (9) in the definitions above might be that those enables us to determine the denotation of a λ-abstraction $\lambda x^\sigma.M$ uniquely based on its extensional behaviour; namely, as the unique element satisfying (7). This makes the presentation of term-interpretation considerably simpler, which is actually presented by mere induction on the structure of λ-terms.

By contrast, in this paper we focus our attention to a weaker variant of semantical frameworks in which a term-interpretation only satisfying the equalities (1)-(6) plus the following conditions:

$$(10) \qquad M =_\beta N \Longrightarrow \forall \xi \quad [\![M]\!]_\xi = [\![N]\!]_\xi$$

We call the structures satisfying these requirements *models* of simply typed λ-calculus. Note that (7) is satisfied in every model.

It is well-known that in every weak-extensional model (9) can be replaced with each of the following condition:

$$(11) \qquad \forall \xi \quad [\![\lambda x^{\sigma\to\tau} y^\sigma.x^{\sigma\to\tau} y^\sigma]\!]_\xi = [\![\lambda x^{\sigma\to\tau}.x^{\sigma\to\tau}]\!]_\xi$$

$$(12) \qquad x^\sigma \notin \mathrm{FV}(M) \Longrightarrow \forall \xi \quad [\![\lambda x^\sigma.Mx^\sigma]\!]_\xi = [\![M]\!]_\xi$$

In this respect, we note that (11) implies (12) even in every model. However the converse does not hold in general, a counterexample of which we are actually to present at the end of this paper.

3. A free category of semi-CCC operations

For giving our categorical framework to model simple types, we begin with a graph underlying it. Let us first consider a sequence $A_0, A_1, A_2, \dots$ of non-empty sets. Then, objects of this graph are given by the sets each of which is denoted by $[\tau]$ for some simple type τ and generated in conjunction with arrows by the following simultaneous induction:

$$\begin{aligned}
&a \in A_i \Longrightarrow a \in [\alpha_i],\\
&* \in [1],\\
&a \in [\sigma] \ \ \& \ \ b \in [\tau] \Longrightarrow (a,b) \in [\sigma \times \tau],\\
&s \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau]) \Longrightarrow s \in [\sigma \to \tau],
\end{aligned}$$

$$
\begin{array}{ll}
\mathrm{id}_{[\sigma]} \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\sigma]), & \bigcirc_{[\sigma]} \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[1]), \\
\mathrm{p}_{[\sigma],[\tau]} \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma\times\tau],[\sigma]), & \mathrm{q}_{[\sigma],[\tau]} \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma\times\tau],[\tau]),
\end{array}
$$

$$\mathrm{ev}_{[\sigma],[\tau]} \in \mathrm{Hom}_{\mathscr{F}_G}([(\tau\to\sigma)\times\tau],[\sigma]),$$

$$a \in [\sigma] \Longrightarrow *\searrow a \in \mathrm{Hom}_{\mathscr{F}_G}([1],[\sigma]),$$

$$s \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau]) \ \ \& \ \ t \in \mathrm{Hom}_{\mathscr{F}_G}([\tau],[\upsilon]) \Longrightarrow t\circ s \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\upsilon]),$$

$$s \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau]) \ \ \& \ \ t \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\upsilon]) \Longrightarrow \langle s,t\rangle \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau\times\upsilon]),$$

$$s \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma\times\tau],[\upsilon]) \Longrightarrow \mathrm{Cur}(s) \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau\to\upsilon]).$$

Here, we designate the set of arrows from $[\sigma]$ to $[\tau]$ by $\mathrm{Hom}_{\mathscr{F}_G}([\sigma],[\tau])$. We write $\mathscr{F}_G$ for the graph so obtained. According to the conventional notation, we also write $\mathrm{Ob}(\mathscr{F}_G)$ for the set of its objects and $\mathrm{Ar}(\mathscr{F}_G)$ for the set of its arrows. We also denote the set of members of all objects in $\mathscr{F}_G$ by T; namely,

$$\mathrm{T} = \bigcup\{[\sigma] \mid \sigma \text{ is a simple type}\}.$$

Note that $\mathrm{Ar}(\mathscr{F}_G) \subseteq \mathrm{T}$. We use letters $s,t,u,v,\ldots$ as meta-variables to designate elements of $\mathrm{Ar}(\mathscr{F}_G)$, and $a,b,\ldots$ to designate elements of T. Unless it does not cause confusion, we drop the information of objects appearing in atomic arrows.

We next establish a notion of application for the arrows of this graph, considering an extension of the graph $\mathscr{F}_G$. It is defined by the same way as $\mathscr{F}_G$ except that we denote each object by $[\sigma]^*$ for some simple type σ, and that we add the following rule for generating its elements:

$$\frac{a \in [\sigma\to\tau]^* \qquad b \in [\sigma]^*}{a(b) \in [\tau]^*}$$

We designate this extended version of graph by $\mathscr{F}_G^*$, and denote the set of members of objects in $\mathscr{F}_G^*$ by T^*; namely,

$$\mathrm{T}^* = \bigcup\{[\sigma]^* \mid \sigma \text{ is a simple type}\}.$$

By the expression $s(a)$ in particular, we intend to describe the result of application of an arrow s to a member a of its domain, which would inevitably lead us to a notion of reduction for evaluation. To be more precise, we consider the smallest binary relation $\rightsquigarrow$ on T^* satisfying:

$$
\begin{array}{ll}
\mathrm{id}(a) \rightsquigarrow a, & \bigcirc(a) \rightsquigarrow *, \\
\mathrm{p}(a,b) \rightsquigarrow a, & (s\circ t)(a) \rightsquigarrow s(t(a)), \\
\mathrm{q}(a,b) \rightsquigarrow b, & \langle s,t\rangle(a) \rightsquigarrow (s(a),t(a)),
\end{array}
$$

$$\mathrm{ev}(a,b) \rightsquigarrow a(b), \qquad \mathrm{Cur}(s)(a) \rightsquigarrow s \circ \langle\, (*\searrow a) \circ \bigcirc, \mathrm{id}\,\rangle,$$
$$(*\searrow a)(*) \rightsquigarrow a,$$

$$\frac{a \rightsquigarrow b}{(a,c) \rightsquigarrow (b,c)} \qquad \frac{a \rightsquigarrow b}{(c,a) \rightsquigarrow (c,b)}$$

$$\frac{a \rightsquigarrow b}{a(c) \rightsquigarrow b(c)} \qquad \frac{a \rightsquigarrow b}{c(a) \rightsquigarrow c(b)}$$

$$\frac{s \rightsquigarrow t}{u \circ s \rightsquigarrow u \circ t} \qquad \frac{s \rightsquigarrow t}{s \circ u \rightsquigarrow t \circ u}$$

$$\frac{s \rightsquigarrow t}{\langle s,u\rangle \rightsquigarrow \langle t,u\rangle} \qquad \frac{s \rightsquigarrow t}{\langle u,s\rangle \rightsquigarrow \langle u,t\rangle}$$

$$\frac{s \rightsquigarrow t}{\mathrm{Cur}(s) \rightsquigarrow \mathrm{Cur}(t)} \qquad \frac{a \rightsquigarrow b}{*\searrow a \rightsquigarrow *\searrow b}$$

According to the usual convention, we designate the reflexive closure of this relation by $\rightsquigarrow^=$, the transitive closure by $\rightsquigarrow^+$ and the reflexive transitive closure by $\rightsquigarrow^*$.

This term rewriting system turns out to be complete. Actually, it is immediate that the reduction $\rightsquigarrow$ does not yield any divergent critical pair and thus satisfies weak Church-Rosser property.

Contrary to this, it seems rather difficult to confirm that no infinite reduction sequence arises under any reduction strategy. To see it, let us suppose that

$$\mathrm{SN} = \{a \in \mathrm{T}^* \mid a \text{ is strongly normalizable}\},$$
$$\mathrm{SN}^{\rightarrow} = \{s(a_0)\cdots(a_n) \in \mathrm{T}^* \mid s \in \mathrm{Ar}(\mathscr{F}_G^*), n \in \mathbb{N} \text{ and } a_0,\ldots,a_n \in \mathrm{SN}\},$$

and that $\nu(a)$ is the length of a longest reduction path starting from a member a of SN. To show the equality $\mathrm{SN} = \mathrm{T}^*$, we still need to consider a restricted version of the reduction, under which we are allowed to reduce the redex only in the head position. Specifically, we define a reduction $\rightsquigarrow_h$ as the smallest relation satisfying the same axioms as $\rightsquigarrow$ and the following rules:

$$\frac{\forall c,d\;\; a \neq (c,d) \qquad a \rightsquigarrow_h b}{\mathrm{p}(a) \rightsquigarrow_h \mathrm{p}(b)} \qquad \frac{\forall c,d\;\; a \neq (c,d) \qquad a \rightsquigarrow_h b}{\mathrm{q}(a) \rightsquigarrow_h \mathrm{q}(b)}$$

$$\frac{\forall c,d\;\; a \neq (c,d) \qquad a \rightsquigarrow_h b}{\mathrm{ev}(a) \rightsquigarrow_h \mathrm{ev}(b)} \qquad \frac{b \neq * \qquad b \rightsquigarrow_h c}{(*\searrow a)(b) \rightsquigarrow_h (*\searrow a)(c)}$$

$$\frac{a \rightsquigarrow_h b}{a(c) \rightsquigarrow_h b(c)}$$

Under these preparations, we can now demonstrate our proof of strong normalizability.

Lemma 1. (i) *If $s \rightsquigarrow_h t_1$ and $s \rightsquigarrow t_2$, then there exists a term t_3 such that $t_1 \rightsquigarrow^* t_3$ and $t_2 \rightsquigarrow_h^= t_3$.*
(ii) *If $s \rightsquigarrow_h t$ where $s \in \mathrm{SN}^{\rightarrow}$ and $t \in \mathrm{SN}$, then $s \in \mathrm{SN}$.*

Proof. (i) We show the statement by induction on the structure of s, and distinguish cases on the generation of $s \rightsquigarrow_h t_1$. We study some principal cases below, in which we do not study the case where $t_1 = t_2$ since we can adopt itself as t_3:

Case 1: Suppose $\mathrm{p}(a, b) \rightsquigarrow_h a$. Then a possible form, except a, of t_2 is either $\mathrm{p}(a', b)$ or $\mathrm{p}(a, b')$ where $a \rightsquigarrow a'$ and $b \rightsquigarrow b'$. Thus, we can set $t_3 = a'$ for the former and $t_3 = a$ for the latter.

Case 2: Suppose $\langle u, v\rangle(a) \rightsquigarrow_h (u(a), v(a))$. Then a possible form, except $(u(a), v(a))$, of t_2 is either $\langle u', v\rangle(a)$, $\langle u, v'\rangle(a)$ or $\langle u, v\rangle(a')$ where $u \rightsquigarrow u'$, $v \rightsquigarrow v'$ and $a \rightsquigarrow a'$. As for them, we can adopt $(u'(a), v(a)), (u(a), v'(a))$ and $(u(a'), v(a'))$ as t_3, respectively.

Case 3: Suppose $\mathrm{Cur}(t)(a) \rightsquigarrow_h t \circ \langle\, (* \searrow a) \circ \bigcirc, \mathrm{id}\,\rangle$. Then a possible form, except $t \circ \langle\, (* \searrow a) \circ \bigcirc, \mathrm{id}\,\rangle$, of t_2 is either $\mathrm{Cur}(t')(a)$ or $\mathrm{Cur}(t)(a')$ where $t \rightsquigarrow t'$ and $a \rightsquigarrow a'$. Thus, we can set $t_3 = t' \circ \langle\, (* \searrow a) \circ \bigcirc, \mathrm{id}\,\rangle$ for the former and $t_3 = t \circ \langle\, (* \searrow a') \circ \bigcirc, \mathrm{id}\,\rangle$ for the latter.

Case 4: Suppose $\mathrm{p}(a) \rightsquigarrow_h \mathrm{p}(b_1)$ where a is not a pair and $a \rightsquigarrow_h b_1$. Then a possible form, except $\mathrm{p}(b_1)$, of t_2 is $\mathrm{p}(b_2)$ where $a \rightsquigarrow b_2 \neq b_1$. Here, whether b_1 is of the form (c, d) or not, we know that b_2 is not in the form of pair; indeed, it is unable to reduce a to a pair by $\rightsquigarrow$ when $a \not\rightsquigarrow_h (c, d)$ and to a pair other than (c, d) when $a \rightsquigarrow_h (c, d)$. Thus, applying the induction hypothesis, we can find a term b_3 such that $b_1 \rightsquigarrow^* b_3$ and $b_2 \rightsquigarrow_h^= b_3$. Therefore we can set $t_3 = \mathrm{p}(b_3)$.

Case 5: Suppose $(* \searrow a)(b) \rightsquigarrow_h (* \searrow a)(c_1)$ where $b \rightsquigarrow_h c_1$. Then a possible form, except $(* \searrow a)(c_1)$, of t_2 is either $(* \searrow a')(b)$ or $(* \searrow a)(c_2)$ where $a \rightsquigarrow a'$ and $b \rightsquigarrow c_2 \neq c_1$. The former case does not yield any difficulty; indeed, we can set $t_3 = (* \searrow a')(c_1)$. In the latter case, even if $c_1 \neq *$, we know that $c_2 \neq *$; indeed, it is unable to have $b \rightsquigarrow *$ when $b \not\rightsquigarrow_h *$. Thus, applying the induction hypothesis, we can find a term c_3 such that $c_1 \rightsquigarrow^* c_3$ and $c_2 \rightsquigarrow_h^= c_3$. Therefore we can set $t_3 = (* \searrow a)(c_3)$.

Case 6: Suppose $u(a) \rightsquigarrow_h v_1(a)$ where $u \rightsquigarrow_h v_1$. Then a possible form, except $v_1(a)$, of t_2 is either $v_2(a)$ or $u(a')$ where $u \rightsquigarrow v_2 \neq v_1$ and $a \rightsquigarrow a'$. In the former case, applying the induction hypothesis, we can find a term v_3 such that $v_1 \rightsquigarrow^* v_3$ and $v_2 \rightsquigarrow_h^= v_3$; thus we can set $t_3 = v_3(a)$. In the latter case, we can set $t_3 = v_1(a')$.

(ii) Note that, from the assumption, we may describe $s = u(a_0) \cdots (a_n)$ for some $u \in \mathrm{Ar}(\mathscr{F}_G^*)$, $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathrm{SN}$. So we show the statement by induction on the *degree*

$$\nu(t) + \sum_{i=0}^{n} \nu(a_i)$$

of the reduction step $s \rightsquigarrow t$. It suffice to prove that terms, except t, obtained from $s = u(a_0) \cdots (a_n)$ by one step reduction on $\rightsquigarrow$ are all strongly normalizable, and we exhibit proofs of some principal cases in the following:

Case 1: Suppose $\mathrm{ev}(a, b)(a_1) \cdots (a_n) \rightsquigarrow_h a(b)(a_1) \cdots (a_n)$. Then we obtain the strong normalizability of $\mathrm{ev}(a', b)(a_1) \cdots (a_n)$ if $a \rightsquigarrow a'$. This is because degree of the reduction step $\mathrm{ev}(a', b)(a_1) \cdots (a_n) \rightsquigarrow_h a'(b)(a_1) \cdots (a_n)$ is strictly smaller than that of the original one, and we can apply the induction hypothesis. Analogously, $\mathrm{ev}(a, b')(a_1) \cdots (a_n)$ and $\mathrm{ev}(a, b)(a_1) \cdots (a'_i) \cdots (a_n)$ are both shown to be strongly normalizable if $b \rightsquigarrow b'$ and $a_i \rightsquigarrow a'_i$.

Case 2: Suppose $(* \searrow a)(b)(a_1) \cdots (a_n) \rightsquigarrow_h (* \searrow a)(b')(a_1) \cdots (a_n)$ where $b \rightsquigarrow_h b'$. We first concentrate on a reduct of the form $(* \searrow a)(c)(a_1) \cdots (a_n)$ where $b \rightsquigarrow c \neq b'$. In this case, even if $b' \neq *$, we know that $c \neq *$, as is observed in the proof of (i). So (i) guarantees the existence of a term c' such that $c \rightsquigarrow_{\bar{h}}^{=} c'$ and $b' \rightsquigarrow^* c'$. Here, we obtain the strong normalizability of $(* \searrow a)(c)(a_1) \cdots (a_n)$ even though $c \neq c'$. This is because degree of the reduction step $(* \searrow a)(c)(a_1) \cdots (a_n) \rightsquigarrow_h (* \searrow a)(c')(a_1) \cdots (a_n)$ is strictly smaller than that of the original one, for which we can apply induction hypothesis. The other reducts we need to verify are either of the form $(* \searrow a')(b)(a_1) \cdots (a_n)$ or $(* \searrow a)(b)(a_1) \cdots (a'_i) \cdots (a_n)$ where $a \rightsquigarrow a'$ and $a_i \rightsquigarrow a'_i$. Their strong normalizability are obtained straightforwardly by means of the induction hypothesis. $\square$

The proof of the strong normalizability which we present here uses a notion of computability predicate. For every $[\sigma]^* \in \mathrm{Ob}(\mathscr{F}_G^*)$, we define a set $\mathrm{Comp}([\sigma]^*)$ by induction on the structure of σ, as follows:

$$
\begin{aligned}
\mathrm{Comp}([\alpha]^*) &= \{a \in [\alpha]^* \mid a \text{ is strongly normalizable}\} \\
\mathrm{Comp}([1]^*) &= \mathrm{Comp}_0([1]^*) \cup \mathrm{Comp}_1([1]^*) \\
\mathrm{Comp}([\sigma \times \tau]^*) &= \mathrm{Comp}_0([\sigma \times \tau]^*) \cup \mathrm{Comp}_1([\sigma \times \tau]^*) \\
\mathrm{Comp}([\sigma \to \tau]^*) &= \{a \in [\sigma \to \tau]^* \mid \forall b \in \mathrm{Comp}([\sigma]^*)\ \ a(b) \in \mathrm{Comp}([\tau]^*)\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Comp}_0([1]^*) &= \{*\} \\
\mathrm{Comp}_1([1]^*) &= \{a \in \mathrm{SN}^{\to} \mid \exists n \in \mathbb{N}\ \exists a_1, \ldots, a_n \in \mathrm{SN}^{\to} \\
&\qquad\qquad a \rightsquigarrow_h a_1 \rightsquigarrow_h \cdots \rightsquigarrow_h a_n \rightsquigarrow_h *\}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Comp}_0([\sigma \times \tau]^*) &= \{(a, b) \in [\sigma \times \tau]^* \mid a \in \mathrm{Comp}([\sigma]^*)\ \ b \in \mathrm{Comp}([\tau]^*)\} \\
\mathrm{Comp}_1([\sigma \times \tau]^*) &= \{a \in \mathrm{SN}^{\to} \mid \exists n \in \mathbb{N}\ \exists a_1, \ldots, a_n \in \mathrm{SN}^{\to} \\
&\qquad\qquad \exists b \in \mathrm{Comp}([\sigma]^*)\ \exists c \in \mathrm{Comp}([\tau]^*) \\
&\qquad\qquad a \rightsquigarrow_h a_1 \rightsquigarrow_h \cdots \rightsquigarrow_h a_n \rightsquigarrow_h (b, c)\}
\end{aligned}
$$

Lemma 2. *The following hold for every* $[\sigma]^* \in \mathrm{Ob}(\mathscr{F}_G^*)$:

(i) $\mathrm{Comp}([\sigma]^*) \neq \emptyset$.

(ii) *If* $a \in \mathrm{Comp}([\sigma]^*)$, *then* $a \in \mathrm{SN}$.

(iii) *If* $a \rightsquigarrow_h b$ *where* $a \in \mathrm{SN}^{\to}$ *and* $b \in \mathrm{Comp}([\sigma]^*)$, *then* $a \in \mathrm{Comp}([\sigma]^*)$.

Proof. We simultaneously verify all of the statements above by induction on the structure of σ:

(i) Suppose σ is of the form $\tau \to \upsilon$. Then we can find a member $c \in \mathrm{Comp}([\upsilon]^*)$ by the induction hypothesis of (i), and

$$
\begin{aligned}
((* \searrow c) \circ \bigcirc)(b) &\rightsquigarrow_h (* \searrow c)(\bigcirc(b)) \\
&\rightsquigarrow_h (* \searrow c)(*) \\
&\rightsquigarrow_h c
\end{aligned}
$$

holds for every $b \in \mathrm{Comp}([\tau]^*)$. Thus, applying the induction hypotheses of (ii) and (iii), we obtain $((* \searrow c) \circ \bigcirc)(b) \in \mathrm{Comp}([\upsilon]^*)$. As a result, we conclude $(* \searrow c) \circ \bigcirc \in \mathrm{Comp}([\tau \to \upsilon]^*)$. Proofs for the other cases are immediate, which we omit.

(ii) If σ is a type constant other than 1, then the statement is clear from the definition. So we confirm the other cases below:

Case 1: Suppose $a = (b, c) \in \mathrm{Comp}_0([\tau \times \upsilon]^*)$, which entails $b \in \mathrm{Comp}([\tau]^*)$ and $c \in \mathrm{Comp}([\upsilon]^*)$. Then $b, c \in \mathrm{SN}$, and so $(b, c) \in \mathrm{SN}$, follows by the induction hypothesis of (ii).

Case 2: Suppose $a \in \mathrm{Comp}_1([\tau \times \upsilon]^*)$; that is,

$$
a \rightsquigarrow_h a_1 \rightsquigarrow_h \cdots \rightsquigarrow_h a_n \rightsquigarrow_h (b, c)
$$

for some $a_1, \ldots, a_n \in \mathrm{SN}^{\to}$, $b \in \mathrm{Comp}([\tau]^*)$ and $c \in \mathrm{Comp}([\upsilon]^*)$. Then we obtain $(b, c) \in \mathrm{SN}$ as the preceding case. Thus consecutive applications of Lemma 1 (ii) completes the proof of this case. We can also verify the case where $a \in \mathrm{Comp}_1([1]^*)$ likewise.

Case 3: Suppose $a \in \mathrm{Comp}([\tau \to \upsilon]^*)$. By virtue of the induction hypothesis of (i), we can find at least one element, say b, in $\mathrm{Comp}([\tau]^*)$. Thus $a(b) \in \mathrm{Comp}([\upsilon]^*)$, from which $a(b) \in \mathrm{SN}$ follows by induction hypothesis of (ii). This entails $a \in \mathrm{SN}$.

(iii) Suppose $b \in \mathrm{Comp}([\tau \to \upsilon]^*)$ and $c \in \mathrm{Comp}([\tau]^*)$. Then $a(c) \rightsquigarrow_h b(c)$ and $b(c) \in \mathrm{Comp}([\upsilon]^*)$ are clear from the definition. Here $a(c) \in \mathrm{SN}^{\to}$ holds because $c \in \mathrm{SN}$ by induction hypothesis of (ii). Thus $a(c) \in \mathrm{Comp}([\upsilon]^*)$ by induction hypothesis of (iii). So we conclude $a \in \mathrm{Comp}([\tau \to \upsilon]^*)$. The other cases are ensured by Lemma 1 (ii) and the definition of the computability predicate. $\square$

Lemma 3. (i) *If $a \in [\sigma]^*$, then $a \in \mathrm{Comp}([\sigma]^*)$.*
(ii) *If $s \in \mathrm{Hom}_{\mathscr{F}_G^*}([\sigma]^*, [\tau]^*)$, then $s \in \mathrm{Comp}([\sigma \to \tau]^*)$.*

Proof. By simultaneous induction on the generation of a and s. It is straightforward to confirm (i). Especially, the case where it is a member of an exponential as an arrow of $\mathscr{F}_G^*$ is ensured by the induction hypothesis of (ii). Considering the case where $a(b) \in [\tau]^*$ is induced from $a \in [\sigma \to \tau]^*$ and $b \in [\sigma]^*$, we obtain $a(b) \in \mathrm{Comp}([\tau]^*)$ by induction hypothesis and the definition of the computability predicate.

We then study (ii). As an exemplification concerning base cases, we concentrate on the case where $s = \mathrm{ev}$ as an arrow from $[(\tau \to \sigma) \times \tau]^*$ to $[\sigma]^*$. We let $b \in \mathrm{Comp}_1([(\tau \to \sigma) \times \tau]^*)$. Then we have a reduction sequence

$$\mathrm{ev}(b) \rightsquigarrow_h \cdots \rightsquigarrow_h \mathrm{ev}(c, d) \rightsquigarrow_h c(d)$$

for some $c \in \mathrm{Comp}([\tau \to \sigma]^*)$ and $d \in \mathrm{Comp}([\tau]^*)$. Here note that all terms appearing in this sequence belong to $\mathrm{SN}^{\to}$ by Lemma 2 (ii) and that $c(d) \in \mathrm{Comp}([\sigma]^*)$ follows from the definition of the computability predicate. Thus we obtain $\mathrm{ev}(b) \in \mathrm{Comp}([\sigma]^*)$ by means of Lemma 2 (iii). Likewise, $\mathrm{ev}(c, d) \in \mathrm{Comp}([\sigma]^*)$ holds for every $(c, d) \in \mathrm{Comp}_0([(\tau \to \sigma) \times \tau]^*)$. As a consequence, we obtain

$$\mathrm{ev} \in \mathrm{Comp}([(\tau \to \sigma) \times \tau \to \sigma]^*).$$

Turning our attention to compound terms, we consider the case where $s = \mathrm{Cur}(t)$ as an arrow from $[\sigma]^*$ to $[\tau \to \upsilon]^*$. We let $b \in \mathrm{Comp}([\sigma]^*)$ and $c \in \mathrm{Comp}([\tau]^*)$. Then we have

$$\begin{aligned} \mathrm{Cur}(t)(b)(c) &\rightsquigarrow_h (t \circ \langle (* \searrow b) \circ \bigcirc, \mathrm{id}\rangle)(c) \\ &\rightsquigarrow_h t(\langle (* \searrow b) \circ \bigcirc, \mathrm{id}\rangle(c)). \end{aligned}$$

Here we know $((* \searrow b) \circ \bigcirc)(c) \in \mathrm{Comp}([\sigma]^*)$ by applying Lemma 2 (iii) to the reduction sequence

$$\begin{aligned} ((* \searrow b) \circ \bigcirc)(c) &\rightsquigarrow_h (* \searrow b)(\bigcirc(c)) \\ &\rightsquigarrow_h (* \searrow b)(*) \\ &\rightsquigarrow_h b, \end{aligned}$$

and $\mathrm{id}(c) \in \mathrm{Comp}([\tau]^*)$ likewise. So $\langle (* \searrow b) \circ \bigcirc, \mathrm{id}\rangle(c) \in \mathrm{Comp}([\sigma \times \tau]^*)$ follows, and hence $t(\langle (* \searrow b) \circ \bigcirc), \mathrm{id}\rangle(c)) \in \mathrm{Comp}([\upsilon]^*)$ by the definition of the computability predicate and the induction hypothesis of (ii). This together with Lemma 2 (iii) implies $\mathrm{Cur}(t)(b)(c) \in \mathrm{Comp}([\upsilon]^*)$. This is the reason why

$$\mathrm{Cur}(t) \in \mathrm{Comp}([\sigma \to \tau \to \upsilon]^*).$$

Proofs for the other cases are similar, which we omit. $\square$

Combining Lemma 2 (ii) and Lemma 3 (i), we now obtain the strong normalizability which we expected. Accordingly, it follows from the discussion presented here that every $a \in \mathrm{T}^*$ has unique normal form in T, for which we write $[a]_{\mathrm{N}}$.

Theorem 4. *If $a \in \mathrm{T}^*$, then $a \in \mathrm{SN}$.*

Next we introduce an equality $\sim$ among the elements of T so as to make the graph $\mathscr{F}_G$ a free semi-ccc which we expect. It is defined to be the smallest equivalence relation satisfying

$$\mathrm{ev} \circ \langle (* \searrow a) \circ \bigcirc, \mathrm{id}\rangle \sim a, \tag{13}$$

$$s \circ (* \searrow a) \sim * \searrow [s(a)]_{\mathrm{N}}, \tag{14}$$

as well as the following axioms and rules of semi-ccc:

$$\begin{array}{cc}
\mathrm{id} \circ s \sim s, & s \circ \mathrm{id} \sim s, \\
(s \circ t) \circ u \sim s \circ (t \circ u), & \langle s, t \rangle \circ u \sim \langle s \circ u, t \circ u \rangle, \\
\mathrm{p} \circ \langle s, t \rangle \sim s, & \mathrm{q} \circ \langle s, t \rangle \sim t, \\
\mathrm{ev} \circ \langle \mathrm{Cur}(t) \circ u, v \rangle \sim t \circ \langle u, v \rangle, & \mathrm{Cur}(s) \circ t \sim \mathrm{Cur}(s \circ \langle t \circ \mathrm{p}, \mathrm{q} \rangle),
\end{array}$$

$$\begin{array}{cc}
\dfrac{a \sim b}{(a, c) \sim (b, c)} & \dfrac{a \sim b}{(c, a) \sim (c, b)} \\[2ex]
\dfrac{s \sim t}{s \circ u \sim t \circ u} & \dfrac{s \sim t}{u \circ s \sim u \circ t} \\[2ex]
\dfrac{s \sim t}{\langle s, u \rangle \sim \langle t, u \rangle} & \dfrac{s \sim t}{\langle u, s \rangle \sim \langle u, t \rangle} \\[2ex]
\dfrac{s \sim t}{\mathrm{Cur}(s) \sim \mathrm{Cur}(t)} & \dfrac{a \sim b}{* \searrow a \sim * \searrow b}
\end{array}$$

As usual, we are to model λ-terms as arrows of our introducing semi-ccc. Thus, from the viewpoint of our main purpose to discard the weak-extensionality from our semantics, it is essential that extensionality of the application of arrows is not established modulo the equality $\sim$. To ensure it, we need allow the existence of $s, t \in \mathrm{Ar}(\mathscr{F}_G)$ such that $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$ holds whereas $s \not\sim t$. In this respect, we require the equality of ccc under the equality of extensional collapse. Indeed, incorporating (13) allows us to identify $[\mathrm{Cur}(\mathrm{ev})(a)]_{\mathrm{N}}$ and $[\mathrm{id}(a)]_{\mathrm{N}}$ for every a. We also adopt (14) in our definition, which implies the well-definedness of the application of arrows modulo the equality $\sim$.

Lemma 5. *For every $s, t \in \mathrm{Hom}_{\mathscr{F}_G}([\sigma], [\tau])$ and $a, b \in [\sigma]$, $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$ follows whenever $s \sim t$ and $a \sim b$.*

Proof. By induction on the number $\nu(s(a)) + \nu(t(b))$. We distinguish cases on the generation of $s \sim t$:

Case 1: Suppose $s = \mathrm{id} \circ u$ and $t = u$. Then we have $\nu(u(a)) < \nu(s(a))$ since $s(a) \rightsquigarrow^+ u(a)$, so that $[s(a)]_{\mathrm{N}} = [u(a)]_{\mathrm{N}} \sim [u(b)]_{\mathrm{N}}$ follows by the induction hypothesis.

Case 2: Suppose $s = u \circ (v \circ w)$ and $t = (u \circ v) \circ w$. Then we have

$$\nu(w(a)) < \nu(s(a)) \quad \text{and} \quad \nu(w(b)) < \nu(t(b))$$

because $s(a) \rightsquigarrow^+ u(v(w(a)))$ and $t(b) \rightsquigarrow^+ u(v(w(b)))$. Therefore we obtain $[w(a)]_{\mathrm{N}} \sim [w(b)]_{\mathrm{N}}$ by induction hypothesis. Furthermore this implies $[v([w(a)]_{\mathrm{N}})]_{\mathrm{N}} \sim [v([w(b)]_{\mathrm{N}})]_{\mathrm{N}}$ by induction hypothesis, so that

$$[v(w(a))]_{\mathrm{N}} \sim [v(w(a))]_{\mathrm{N}}.$$

This is because we have $\nu(v([w(a)]_{\mathrm{N}})) < \nu(s(a))$ from $s(a) \rightsquigarrow^+ u(v([w(a)]_{\mathrm{N}}))$, and $\nu(v([w(b)]_{\mathrm{N}})) < \nu(t(b))$ likewise. By applying the same discussion again, we obtain $[u([v(w(a))]_{\mathrm{N}})]_{\mathrm{N}} \sim [u([v(w(b))]_{\mathrm{N}})]_{\mathrm{N}}$ and conclude $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$.

Case 3: Suppose $s = \langle u, v\rangle \circ w$ and $t = \langle u \circ w, v \circ w\rangle$. Then we have

$$\nu(w(a)) < \nu(s(a)) \quad \text{and} \quad \nu(w(b)) < \nu(t(b))$$

because $s(a) \rightsquigarrow^+ (u(w(a)), v(w(a)))$ and $t(b) \rightsquigarrow^+ (u(w(b)), v(w(b)))$. Therefore we obtain $[w(a)]_{\mathrm{N}} \sim [w(b)]_{\mathrm{N}}$ by induction hypothesis. Furthermore this implies $[u([w(a)]_{\mathrm{N}})]_{\mathrm{N}} \sim [u([w(b)]_{\mathrm{N}})]_{\mathrm{N}}$ by induction hypothesis, so that

$$[u(w(a))]_{\mathrm{N}} \sim [u(w(b))]_{\mathrm{N}}$$

follows. This is because we have $\nu(u([w(a)]_{\mathrm{N}})) < \nu(s(a))$ from $s(a) \rightsquigarrow^+ (u([w(a)]_{\mathrm{N}}), v([w(a)]_{\mathrm{N}}))$, and $\nu(u([w(b)]_{\mathrm{N}})) < \nu(t(b))$ likewise. We can also verify $[v(w(a))]_{\mathrm{N}} \sim [v(w(b))]_{\mathrm{N}}$ by the same discussion. Hence we obtain $([u(w(a))]_{\mathrm{N}}, [v(w(a))]_{\mathrm{N}}) \sim ([u(w(b))]_{\mathrm{N}}, [v(w(b))]_{\mathrm{N}})$, and conclude $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$.

Case 4: Suppose $s = \mathrm{p} \circ \langle u, v\rangle$ and $t = u$. Then we have $\nu(u(a)) < \nu(s(a))$ since $s(a) \rightsquigarrow^+ u(a)$, so that $[s(a)]_{\mathrm{N}} = [u(a)]_{\mathrm{N}} \sim [u(b)]_{\mathrm{N}}$ follows by the induction hypothesis.

Case 5: Suppose $s = \mathrm{ev} \circ \langle \mathrm{Cur}(u) \circ \mathrm{p}, \mathrm{q}\rangle$ and $t = u \circ \langle \mathrm{p}, \mathrm{q}\rangle$. Then we have $\nu(\mathrm{p}(a)) < \nu(s(a))$ and $\nu(\mathrm{p}(b)) < \nu(t(b))$ because $s(a) \rightsquigarrow^+ u(\mathrm{p}(a), \mathrm{q}(a))$ and $t(b) \rightsquigarrow^+ u(\mathrm{p}(b), \mathrm{q}(b))$. Therefore we obtain $[\mathrm{p}(a)]_{\mathrm{N}} \sim [\mathrm{p}(b)]_{\mathrm{N}}$ by induction hypothesis. Likewise, we also obtain $[\mathrm{q}(a)]_{\mathrm{N}} \sim [\mathrm{q}(b)]_{\mathrm{N}}$, from which

$$[(\mathrm{p}(a), \mathrm{q}(a))]_{\mathrm{N}} \sim [(\mathrm{p}(b), \mathrm{q}(b))]_{\mathrm{N}}$$

follows immediately. This implies $[u([(\mathrm{p}(a), \mathrm{q}(a))]_{\mathrm{N}})]_{\mathrm{N}} \sim [u([(\mathrm{p}(b), \mathrm{q}(b))]_{\mathrm{N}})]_{\mathrm{N}}$ by induction hypothesis, so that

$$[u(\mathrm{p}(a), \mathrm{q}(a))]_{\mathrm{N}} \sim [u(\mathrm{p}(b), \mathrm{q}(b))]_{\mathrm{N}}$$

follows. This is because we have $\nu(u([(\mathrm{p}(a), \mathrm{q}(a))]_{\mathrm{N}})) < \nu(s(a))$ from $s(a) \rightsquigarrow^+ u([(\mathrm{p}(a), \mathrm{q}(a))]_{\mathrm{N}})$, and $\nu(u([(\mathrm{p}(b), \mathrm{q}(b))]_{\mathrm{N}})) < \nu(t(b))$ likewise. Therefore we conclude $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$.

Case 6: Suppose $s = \mathrm{Cur}(u) \circ v$ and $t = \mathrm{Cur}(u \circ \langle v \circ \mathrm{p}, \mathrm{q}\rangle)$. Then the equation is shown by

$$\begin{aligned}
[s(a)]_{\mathrm{N}} &= u \circ \langle (* \searrow [v(a)]_{\mathrm{N}}) \circ \bigcirc, \mathrm{id}\rangle \\
&\sim u \circ \langle (v \circ (* \searrow a)) \circ \bigcirc, \mathrm{id}\rangle \\
&\sim (u \circ \langle v \circ \mathrm{p}, \mathrm{q}\rangle) \circ \langle (* \searrow b) \circ \bigcirc, \mathrm{id}\rangle \\
&= [t(b)]_{\mathrm{N}}.
\end{aligned}$$

Case 7: Suppose $s = u \circ (* \searrow c)$ and $t = * \searrow [u(c)]_{\mathrm{N}}$, which inevitably entails that $a = b = *$. Hence we obtain $[s(a)]_{\mathrm{N}} = [u(c)]_{\mathrm{N}} = [t(b)]_{\mathrm{N}}$.

Case 8: Suppose $s = t$. Then we further distinguish cases on the structure of the arrow. Here we concentrate on the case where $s = t = \mathrm{ev}$. Without loss of generality, we may assume that $a = (u, c)$ and $b = (v, d)$ for some $u, v, c, d \in \mathrm{T}$ such that $u \sim v$ and $c \sim d$, for which $\nu(u(c)) < \nu(s(a))$

and $\nu(v(d)) < \nu(t(b))$ is clear. Therefore we obtain $[u(c)]_{\mathrm{N}} \sim [v(d)]_{\mathrm{N}}$ by induction hypothesis, so that $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$. Proofs for the other cases are easier, which we omit.

Case 9: Suppose $s = u \circ v$ and $t = u \circ w$ where $v \sim w$. Then we have $\nu(v(a)) < \nu(s(a))$ and $\nu(w(b)) < \nu(t(b))$ because $s(a) \rightsquigarrow^{+} u(v(a))$ and $t(b) \rightsquigarrow^{+} u(w(b))$. Therefore we obtain

$$[v(a)]_{\mathrm{N}} \sim [w(b)]_{\mathrm{N}}$$

by induction hypothesis. This further implies $[u([v(a)]_{\mathrm{N}})]_{\mathrm{N}} \sim [u([w(b)]_{\mathrm{N}})]_{\mathrm{N}}$ by induction hypothesis, so that

$$[u(v(a))]_{\mathrm{N}} \sim [u(w(b))]_{\mathrm{N}}.$$

This is because we have $\nu(u([v(a)]_{\mathrm{N}})) < \nu(s(a))$ from $s(a) \rightsquigarrow^{+} u([v(a)]_{\mathrm{N}})$, and $\nu(u([w(b)]_{\mathrm{N}})) < \nu(t(b))$ likewise. Hence we obtain $[s(a)]_{\mathrm{N}} \sim [t(b)]_{\mathrm{N}}$. $\square$

Under these preparations, we now present a semi-ccc naively by identifying all of the components of the graph $\mathscr{F}_G$ modulo the equality $\sim$. Indeed, we denoting the semi-ccc by $\mathscr{F}_C$, its objects are exactly quotient sets of objects of $\mathscr{F}_G$ by the equality $\sim$; that is, letting $[\sigma]^{\mathrm{c}}$ be $[\sigma]\,/\!\sim$ for each simple type σ, we define

$$\mathrm{Ob}(\mathscr{F}_C) = \{[\sigma]^{\mathrm{c}} \mid \sigma \text{ is a simple type}\}.$$

Among such objects, we can find the operations of semi-ccc naturally. Actually, we may adopt $[1]^{\mathrm{c}}$ as a terminal object, and define a cartesian product $\times$ and an exponential $\Rightarrow$ by

$$[\sigma]^{\mathrm{c}} \times [\tau]^{\mathrm{c}} = [\sigma \times \tau]^{\mathrm{c}}$$
$$[\sigma]^{\mathrm{c}} \Rightarrow [\tau]^{\mathrm{c}} = [\sigma \to \tau]^{\mathrm{c}}$$

for each simple type σ, τ. We also identify arrows modulo the equality $\sim$, and define

$$\mathrm{Hom}_{\mathscr{F}_C}([\sigma]^{\mathrm{c}}, [\tau]^{\mathrm{c}}) = \mathrm{Hom}_{\mathscr{F}_G}([\sigma], [\tau])\,/\!\sim$$

for each simple type σ, τ. Note that this set coincides with the exponential $[\sigma]^{\mathrm{c}} \Rightarrow [\tau]^{\mathrm{c}}$. Among the equivalence classes so obtained, we then naively consider the same operations as those on T. For example, letting $\underline{s}$, $\underline{t}$ and $\underline{a}$ be equivalence classes, we define $\underline{s} \circ \underline{t}$ and $* \searrow \underline{a}$ to be the equivalence classes containing $s \circ t$ and $* \searrow a$, respectively. To make the presentation less cumbersome, we notationally identify elements of T with equivalence classes containing them by following the ordinary convention.

4. The model

Based on the category $\mathscr{F}_C$, we are to demonstrate our model of simply typed λ-calculus in which a restraint concerning the weak-extensionality property of term-interpretation is established. It is accomplished by two frameworks of interpretation.

The first one is given naively by following the usual discussion of categorical semantics, which comes out to give a certain fine viewpoint of interpretation. Here we trace its presentation with some proofs so as to confirm that it is enough to satisfy (10) even if the equality of arrows in $\mathscr{F}_C$ is now strictly weaker than that of the ordinary discussion of categorical semantics based on cartesian closed category, such as [5, Chapter 3] and [2, Chapter 8]. We note that this sort of presentation based on weakening the equality of cartesian closed category can be also found in some literatures, such as [4] and [6], which might be indispensable to our present purpose.

Our definition of term-interpretation is given as usual. Actually, we consider a categorical version of interpretation of a λ-term M of type σ, which is relative to finite sequence $\varDelta = x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}$ of variables which contains all free-variables in M and the components of which are distinct each other. For this sequence, we define an object $\times(\varDelta)$ of $\mathscr{F}_C$ by the following induction:

$$\times(\varDelta) = \begin{cases} [1]^{\mathrm{c}} & \text{if } n = 0, \\ \times(\varSigma) \times [\sigma_n]^{\mathrm{c}} & \text{if } n \neq 0 \text{ and } \varSigma = x_1^{\sigma_1}, \ldots, x_{n-1}^{\sigma_{n-1}}. \end{cases}$$

We then associate an arrow $[\![M]\!]_\varDelta$ from $\times(\varDelta)$ to $[\sigma]^{\mathrm{c}}$ as an interpretation of M with respect to $\varDelta$. This is defined by induction on the structure of M, as follows:

$$\begin{aligned}
&[\![x^\sigma]\!]_\varDelta = \mathrm{Proj}_i^{\varDelta} \qquad \text{if } x^\sigma \text{ is the } i \text{ element of } \varDelta, \\
&[\![\mathrm{fst}(M)]\!]_\varDelta = \mathrm{p} \circ [\![M]\!]_\varDelta, \\
&[\![\mathrm{snd}(M)]\!]_\varDelta = \mathrm{q} \circ [\![M]\!]_\varDelta, \\
&[\![(M, N)]\!]_\varDelta = \langle\, [\![M]\!]_\varDelta, [\![N]\!]_\varDelta \,\rangle, \\
&[\![MN]\!]_\varDelta = \mathrm{ev} \circ \langle\, [\![M]\!]_\varDelta, [\![N]\!]_\varDelta \,\rangle, \\
&[\![\lambda x^\sigma . M]\!]_\varDelta = \mathrm{Cur}(\, [\![M[x^\sigma := y^\sigma]]\!]_{\varDelta, y^\sigma}) \qquad \text{where } y^\sigma \text{ is fresh.}
\end{aligned}$$

Here, by the expression $\mathrm{Proj}_i^{\varDelta}$ we denote the arrow, called a *generalised projection*, inductively defined by

$$\mathrm{Proj}_i^{\varDelta} \;=\; \begin{cases} \mathrm{q} & \text{if } i = n, \\ \mathrm{Proj}_i^{\varSigma} \circ \mathrm{p} & \text{if } i \neq n \text{ and } \varSigma = x_1^{\sigma_1}, \ldots, x_{n-1}^{\sigma_{n-1}}. \end{cases}$$

For each $i \in \{1, \ldots, n-1\}$, we write $\varDelta_i$ for the sequence obtained by exchanging i and $i+1$ elements in $\varDelta$, namely, $\varDelta_i = x_1^{\sigma_1}, \ldots, x_{i+1}^{\sigma_{i+1}}, x_i^{\sigma_i}, \ldots, x_n^{\sigma_n}$, and define an arrow $\mathrm{Perm}_i^{\varDelta}$ from $\times(\varDelta)$ to $\times(\varDelta_i)$ by

$$\mathrm{Perm}_i^{\varDelta} = \begin{cases} \langle \langle \mathrm{p} \circ \mathrm{p}, \mathrm{q} \rangle, \mathrm{q} \circ \mathrm{p} \rangle & \text{if } i = n-1, \\ \langle \mathrm{Perm}_i^{\varSigma} \circ \mathrm{p}, \mathrm{q} \rangle & \text{if } i \neq n-1 \text{ and } \varSigma = x_1^{\sigma_1}, \ldots, x_{n-1}^{\sigma_{n-1}}. \end{cases}$$

This arrow is called a *permutation*. Among these general projections and permutations, the equations below hold:

$$\mathrm{Proj}\,{}_{i}^{\Delta} = \mathrm{Proj}\,{}_{i+1}^{\Delta_i} \circ \mathrm{Perm}\,{}_{i}^{\Delta}, \tag{15}$$

$$\mathrm{Proj}\,{}_{i+1}^{\Delta} = \mathrm{Proj}\,{}_{i}^{\Delta_i} \circ \mathrm{Perm}\,{}_{i}^{\Delta}, \tag{16}$$

$$\mathrm{Proj}\,{}_{j}^{\Delta} = \mathrm{Proj}\,{}_{j}^{\Delta_i} \circ \mathrm{Perm}\,{}_{i}^{\Delta} \tag{17}$$

where j equals neither i nor $i+1$. These play a role to compensate the difference among interpretations caused by a choice of finite sequence, which underlies the proof of Lemma 8 (i) saying the invariance of the interpretation under β-equality.

Lemma 6. (i) $[\![M]\!]_{\Delta} = [\![M]\!]_{\Delta_i} \circ \mathrm{Perm}\,{}_{i}^{\Delta}$.
(ii) *If x^σ is not free in M and Δ, then* $[\![M]\!]_{\Delta,x^\sigma} = [\![M]\!]_{\Delta} \circ \mathrm{p}$.

Proof. (i) By induction on the structure of M. If M is a variable, then the equality to show turns out to be either of those listed in (15)-(17). If $M \equiv \lambda x^\sigma.N$, then we have

$$\begin{aligned}
[\![\lambda x^\sigma.N]\!]_{\Delta} &= \mathrm{Cur}([\![N[x^\sigma := y^\sigma]]\!]_{\Delta,y^\sigma}) \\
&= \mathrm{Cur}([\![N[x^\sigma := y^\sigma]]\!]_{\Delta_i,y^\sigma} \circ \mathrm{Perm}\,{}_{i}^{\Delta,y^\sigma}) && \text{by i.h.} \\
&= \mathrm{Cur}([\![N[x^\sigma := y^\sigma]]\!]_{\Delta_i,y^\sigma} \circ \langle \mathrm{Perm}\,{}_{i}^{\Delta} \circ \mathrm{p}, \mathrm{q}\rangle) \\
&= [\![\lambda x^\sigma.N]\!]_{\Delta_i} \circ \mathrm{Perm}\,{}_{i}^{\Delta}.
\end{aligned}$$

Proofs for the other cases concerning pairing and application are easier, which we omit.

(ii) By induction on the structure of M. As (i), we concentrate our attention on the cases of (Var) and (Abs): Suppose $M \equiv x_i^{\sigma_i}$, namely, the i element of Δ. Then $[\![x_i^{\sigma_i}]\!]_{\Delta,x^\sigma} = \mathrm{Proj}\,{}_{i}^{\Delta,x^\sigma} = \mathrm{Proj}\,{}_{i}^{\Delta} \circ \mathrm{p} = [\![x_i^{\sigma_i}]\!]_{\Delta} \circ \mathrm{p}$. If $M \equiv \lambda y^\tau.N$, then we have

$$\begin{aligned}
[\![\lambda y^\tau.N]\!]_{\Delta,x^\sigma} &= \mathrm{Cur}([\![N[y^\tau := z^\tau]]\!]_{\Delta,x^\sigma,z^\tau}) \\
&= \mathrm{Cur}([\![N[y^\tau := z^\tau]]\!]_{\Delta,z^\tau,x^\sigma} \circ \mathrm{Perm}\,{}_{n+1}^{\Delta,x^\sigma,z^\tau}) && \text{by (i)} \\
&= \mathrm{Cur}(([\![N[y^\tau := z^\tau]]\!]_{\Delta,z^\tau} \circ \mathrm{p}) \circ \mathrm{Perm}\,{}_{n+1}^{\Delta,x^\sigma,z^\tau}) && \text{by i.h.} \\
&= \mathrm{Cur}([\![N[y^\tau := z^\tau]]\!]_{\Delta,z^\tau} \circ \langle \mathrm{p} \circ \mathrm{p}, \mathrm{q}\rangle) \\
&= [\![\lambda y^\tau.N]\!]_{\Delta} \circ \mathrm{p}.
\end{aligned}$$

□

Lemma 7. $[\![M]\!]_{\Delta,x^\sigma} \circ \langle \mathrm{id}, [\![N]\!]_{\Delta}\rangle = [\![M[x^\sigma := N]]\!]_{\Delta}$.

Proof. By induction on the structure of M. Here we verify the cases of (Var) and (Abs): Let M be a variable. Then we distinguish cases whether it is x^σ or not. For the former case, we obtain

$$[\![x^\sigma]\!]_{\Delta,x^\sigma} \circ \langle \mathrm{id}, [\![N]\!]_{\Delta}\rangle = \mathrm{q} \circ \langle \mathrm{id}, [\![N]\!]_{\Delta}\rangle = [\![N]\!]_{\Delta}.$$

Assuming the latter, say $M \equiv x_i^{\sigma_i}$, we also have

$$[\![x_i^{\sigma_i}]\!]_{\Delta,x^\sigma} \circ \langle \mathrm{id}, [\![N]\!]_\Delta \rangle = \mathrm{Proj}\,_i^n \circ \mathrm{p} \circ \langle \mathrm{id}, [\![N]\!]_\Delta \rangle = [\![x_i^{\sigma_i}]\!]_\Delta.$$

Suppose $M \equiv \lambda y^\tau.P$. Then, no matter whether $y^\tau \equiv x^\sigma$ or not, we can show the equality, as follows:

$$\begin{aligned}
&[\![\lambda y^\tau.P]\!]_{\Delta,x^\sigma} \circ \langle\, \mathrm{id}, [\![N]\!]_\Delta \rangle \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau]]\!]_{\Delta,x^\sigma,z^\tau}) \circ \langle\, \mathrm{id}, [\![N]\!]_\Delta \rangle \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau]]\!]_{\Delta,x^\sigma,z^\tau} \circ \langle\, \langle\, \mathrm{id}, [\![N]\!]_\Delta \rangle \circ \mathrm{p}, \mathrm{q}\, \rangle) \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau]]\!]_{\Delta,x^\sigma,z^\tau} \circ (\mathrm{Perm}\,_{n+1}^{\Delta,z^\tau,x^\sigma} \circ \langle\, \mathrm{id}, [\![N]\!]_\Delta \circ \mathrm{p} \rangle)) \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau]]\!]_{\Delta,z^\tau,x^\sigma} \circ \langle\, \mathrm{id}, [\![N]\!]_\Delta \circ \mathrm{p} \rangle) && \text{by (i)} \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau]]\!]_{\Delta,z^\tau,x^\sigma} \circ \langle\, \mathrm{id}, [\![N]\!]_{\Delta,z^\tau} \rangle) && \text{by (ii)} \\
&\quad = \mathrm{Cur}([\![P[y^\tau := z^\tau][x^\sigma := N]]\!]_{\Delta,z^\tau}) && \text{by i.h.} \\
&\quad = [\![(\lambda y^\tau.P)[x^\sigma := N]]\!]_\Delta.
\end{aligned}$$

Proofs for the other cases are easier, which we omit. □

Lemma 8. (i)*If $M =_\beta N$ and Δ contains all free-variables appearing either in M or in N, then $[\![M]\!]_\Delta = [\![N]\!]_\Delta$.*
(ii) $[\![\lambda x^{\sigma\to\tau} y^\sigma.x^{\sigma\to\tau} y^\sigma]\!]_\Delta \neq [\![\lambda x^{\sigma\to\tau}.x^{\sigma\to\tau}]\!]_\Delta$.
(iii) $[\![\lambda y^\sigma.My^\sigma]\!]_\Delta \neq [\![M]\!]_\Delta$ *for every M such that $y^\sigma \notin \mathrm{FV}(M)$.*

Proof. (i) By induction on the generation of the equality $M =_\beta N$. Especially, the axiom of β-equality is shown to be satisfied, as follows:

$$\begin{aligned}
[\![(\lambda x^\sigma.M)N]\!]_\Delta &= \mathrm{ev} \circ \langle \mathrm{Cur}(\, [\![M]\!]_{\Delta,x^\sigma}), [\![N]\!]_\Delta \rangle \\
&= \mathrm{ev} \circ (\langle\, \mathrm{Cur}(\, [\![M]\!]_{\Delta,x^\sigma}) \circ \mathrm{p}, \mathrm{q}\, \rangle \circ \langle \mathrm{id}, [\![N]\!]_\Delta \rangle) \\
&= ([\![M]\!]_{\Delta,x^\sigma} \circ \langle \mathrm{p}, \mathrm{q} \rangle) \circ \langle \mathrm{id}, [\![N]\!]_\Delta \rangle \\
&= [\![M]\!]_{\Delta,x^\sigma} \circ \langle \mathrm{id}, [\![N]\!]_\Delta \rangle \\
&= [\![M[x^\sigma := N]]\!]_\Delta && \text{by Lemma 7}
\end{aligned}$$

Proofs for the other cases are easier, which we omit.

(ii) We know that the left-hand side is equal to $\mathrm{Cur}(\mathrm{Cur}(\mathrm{ev}) \circ \mathrm{q})$ but on the other hand the right-hand side is $\mathrm{Cur}(\mathrm{q})$.

(iii) We know that the left-hand side is equal to $\mathrm{Cur}(\mathrm{ev}) \circ [\![M]\!]_\Delta$, which is not identical with $[\![M]\!]_\Delta$ under the equality of semi-ccc. □

The foregoing discussion enables us to consider a typed applicative structure $\mathscr{A} = \langle [\![\;]\!]^{\mathscr{A}}, \mathrm{Fst}, \mathrm{Snd}, \mathrm{Pair}, \mathrm{App} \rangle$ whose components are given by

$$\begin{aligned}
&[\![\sigma]\!]^{\mathscr{A}} = \mathrm{Hom}_{\mathscr{F}_C}([1]^{\mathrm{c}}, [\sigma]^{\mathrm{c}}), \\
&\mathrm{Fst}^{\sigma,\tau}(s) = \mathrm{p} \circ s, \\
&\mathrm{Snd}^{\sigma,\tau}(s) = \mathrm{q} \circ s, \\
&\mathrm{Pair}^{\sigma,\tau}(s,t) = \langle s,t \rangle, \\
&\mathrm{App}^{\sigma,\tau}(s,t) = \mathrm{ev} \circ \langle s,t \rangle.
\end{aligned}$$

Furthermore this together with the following term-interpretation yields a model of simply typed λ-calculus:

$$[\![M]\!]^{\mathscr{A}}_{\xi} = [\![M]\!]_{\Delta} \circ \xi^{\Delta}$$

where, we writing ϵ for the empty sequence, ξ^{Δ} is defined by

$$\xi^{\Delta} = \begin{cases} \bigcirc & \text{if } \Delta = \epsilon, \\ \langle \xi^{\Sigma}, \xi(x^{\sigma}) \rangle & \text{if } \Delta = \Sigma, x^{\sigma}. \end{cases}$$

The model $\mathscr{A}$ so introduced seems to give a certain fine viewpoint of interpretation. Indeed, concentrating on the conditions for η-equality, we have the following evaluation of interpretation:

$$\begin{aligned} [\![\lambda x^{\sigma \to \tau} y^{\sigma}. x^{\sigma \to \tau} y^{\sigma}]\!]^{\mathscr{A}}_{\xi} &= \mathrm{Cur}(\mathrm{Cur}(\mathrm{ev}) \circ \mathrm{q}) \\ [\![\lambda x^{\sigma \to \tau}. x^{\sigma \to \tau}]\!]^{\mathscr{A}}_{\xi} &= \mathrm{Cur}(\mathrm{q}) \\ [\![\lambda y^{\sigma}. M y^{\sigma}]\!]^{\mathscr{A}}_{\xi} &= \mathrm{Cur}(\mathrm{ev}) \circ [\![M]\!]_{\Delta} \circ \xi^{\Delta} \end{aligned}$$

So we know that not only (12) but also (11) is not satisfied in $\mathscr{A}$. Hence it seems hard and opaque at least only from this observation whether (8) is true in $\mathscr{A}$ or not.

This is the main reason why we introduce a new coarse viewpoint of interpretation and a devise to model the syntax of application and free-variables. In this respect, we are indeed able to find an alternative at this point, allowed to consider the typed applicative structure $\mathscr{B} = \langle [\![\ ]\!]^{\mathscr{B}}, \mathrm{Fst}, \mathrm{Snd}, \mathrm{Pair}, \mathrm{App} \rangle$ whose components are given by

$$\begin{aligned} &[\![\sigma]\!]^{\mathscr{B}} = [\sigma]^{\mathrm{c}}, \\ &\mathrm{Fst}^{\sigma,\tau}(a, b) = a, \\ &\mathrm{Snd}^{\sigma,\tau}(a, b) = b, \\ &\mathrm{Pair}^{\sigma,\tau}(a, b) = (a, b), \\ &\mathrm{App}^{\sigma,\tau}(t, a) = [t(a)]_{\mathrm{N}}. \end{aligned}$$

We then define a mapping of term-interpretation by

$$[\![M]\!]^{\mathscr{B}}_{\xi} = [\, [\![M]\!]_{\Delta}(\xi_{\Delta}) \,]_{\mathrm{N}} \tag{18}$$

where

$$\xi_{\Delta} = \begin{cases} * & \text{if } \Delta = \epsilon, \\ (\xi_{\Sigma}, \xi(x^{\sigma})) & \text{if } \Delta = \Sigma, x^{\sigma}. \end{cases}$$

The result of evaluating the right-hand side of (18) is independent from the choice of the sequence Δ, so that the mapping $[\![\ ]\!]^{\mathscr{B}}$ is well-defined. This fact is actually shown by induction on the structure of M; especially the case where M is in the form of λ-abstraction is ensured by proving the following more general equality:

$$\begin{aligned}&\mathbf{[}M\mathbf{]}_{\Delta,x^\sigma,\Gamma}\circ\langle\langle\cdots\langle\langle(*\searrow\xi_\Delta)\circ\bigcirc,\mathrm{id}\rangle\overbrace{\circ\mathrm{p},\mathrm{q}\rangle\cdots\rangle\circ\mathrm{p},\mathrm{q}\rangle}^{n\text{-times}}\\ &\quad=\mathbf{[}M\mathbf{]}_{\Sigma,x^\sigma,\Gamma}\circ\langle\langle\cdots\langle\langle(*\searrow\xi_\Sigma)\circ\bigcirc,\mathrm{id}\rangle\underbrace{\circ\mathrm{p},\mathrm{q}\rangle\cdots\rangle\circ\mathrm{p},\mathrm{q}\rangle}_{n\text{-times}}\end{aligned}$$

where n is the length of Γ. As it is shown in the following theorem, this mapping actually satisfies all requirements for term-interpretation, and together with the structure $\mathscr{B}$ comes out to be a model of simply typed λ-calculus. Furthermore, we know that (12) is satisfied in this model.

Theorem 9. *The structure $\mathscr{B}$ is a model of simply typed λ-calculus in which* (12) *is satisfied but* (11) *is not.*

Proof. (1) is clear from the definition since we obtain $\xi_\Delta=\rho_\Delta$ if we take a sequence Δ such that $\mathrm{FV}(M)=\{x^\sigma|\ x^\sigma$ appears in $\Delta\}$. As for the other conditions, we obtain (2) by $[\![x^\sigma]\!]^{\mathscr{B}}_\xi=[\,\mathbf{[}x^\sigma\mathbf{]}_{x^\sigma}(*,\xi(x^\sigma))\,]_{\mathrm{N}}=[\,\mathrm{q}(*,\xi(x^\sigma))\,]_{\mathrm{N}}=\xi(x^\sigma)$. We can show (3), and (4) analogously, by

$$\begin{aligned}[\![\mathrm{fst}(M)]\!]^{\mathscr{B}}_\xi&=[\,\mathbf{[}\mathrm{fst}(M)\mathbf{]}_\Delta(\xi_\Delta)\,]_{\mathrm{N}}\\&=[\,(\mathrm{p}\circ\mathbf{[}M\mathbf{]}_\Delta)(\xi_\Delta)\,]_{\mathrm{N}}\\&=\mathrm{Fst}^{\sigma,\tau}([\,\mathbf{[}M\mathbf{]}_\Delta(\xi_\Delta)\,]_{\mathrm{N}})\\&=\mathrm{Fst}^{\sigma,\tau}([\![M]\!]^{\mathscr{B}}_\xi).\end{aligned}$$

For (5) and (6), we obtain

$$\begin{aligned}[\![(M,N)]\!]^{\mathscr{B}}_\xi&=[\,\mathbf{[}(M,N)\mathbf{]}_\Delta(\xi_\Delta)\,]_{\mathrm{N}}\\&=[\,\langle\mathbf{[}M\mathbf{]}_\Delta,\mathbf{[}N\mathbf{]}_\Delta\rangle(\xi_\Delta)\,]_{\mathrm{N}}\\&=[\,(\mathbf{[}M\mathbf{]}_\Delta(\xi_\Delta),\mathbf{[}N\mathbf{]}_\Delta(\xi_\Delta))\,]_{\mathrm{N}}\\&=\mathrm{Pair}^{\sigma,\tau}([\![M]\!]^{\mathscr{B}}_\xi,[\![N]\!]^{\mathscr{B}}_\xi)\end{aligned}$$

and

$$\begin{aligned}[\![MN]\!]^{\mathscr{B}}_\xi&=[\,\mathbf{[}MN\mathbf{]}_\Delta(\xi_\Delta)\,]_{\mathrm{N}}\\&=[\,(\mathrm{ev}\circ\langle\mathbf{[}M\mathbf{]}_\Delta,\mathbf{[}N\mathbf{]}_\Delta\rangle)(\xi_\Delta)\,]_{\mathrm{N}}\\&=[\,(\mathbf{[}M\mathbf{]}_\Delta(\xi_\Delta))(\mathbf{[}N\mathbf{]}_\Delta(\xi_\Delta))\,]_{\mathrm{N}}\\&=[\,[\![M]\!]^{\mathscr{B}}_\xi([\![N]\!]^{\mathscr{B}}_\xi)\,]_{\mathrm{N}}\\&=\mathrm{App}^{\sigma,\tau}([\![M]\!]^{\mathscr{B}}_\xi,[\![N]\!]^{\mathscr{B}}_\xi)\end{aligned}$$

respectively. Now (10) is remaining to see the structure $\mathscr{B}$ is a model, which immediately follows from Lemma 8 (i).

Directing our attention to (11) and (12), we have $[\![\lambda x^{\sigma\to\tau}y^\sigma.x^{\sigma\to\tau}y^\sigma]\!]^{\mathscr{B}}_\xi=\mathrm{Cur}(\mathrm{ev})$ and $[\![\lambda x^{\sigma\to\tau}.x^{\sigma\to\tau}]\!]^{\mathscr{B}}_\xi=\mathrm{id}$. They do not coincide under the equality of our semi-ccc, from which (11) is clearly shown to be not satisfied. On the other hand, we have

$$\begin{aligned}[\![\lambda y^\sigma . M y^\sigma]\!]^{\mathscr{B}}_\xi &= (\mathrm{Cur}(\mathrm{ev}) \circ [\![M]\!]_\Delta)(\xi_\Delta) \\ &= \mathrm{ev} \circ \langle (* \searrow [\![M]\!]^{\mathscr{B}}_\xi) \circ \bigcirc, \mathrm{id} \rangle \\ &= [\![M]\!]^{\mathscr{B}}_\xi \end{aligned}$$

for (12) by virtue of (13). □

As a consequence of the discussion above, we eventually know which conditions hold in the models $\mathscr{A}$ and $\mathscr{B}$. It is summarised in the following table:

Model	Satisfied condition	Dissatisfied condition
$\mathscr{A}$	(10)	(11) (12)
$\mathscr{B}$	(10) (12)	(8) (11)

It would be worth mentioning that (8) is not the case in $\mathscr{B}$ as we have expected, which is entailed from the remarkable feature of the model that the difference of the two arrows in Lemma 8 (iii) collapse whenever they are applied to an element of an object. This yields the sharp contrast between term-interpretations in $\mathscr{A}$ and $\mathscr{B}$, and it is definitely caused by our explicitly demanding (13) in the definition of semi-ccc. The discussion studied in this paper leads us to a way of explicitly manipulating information on the weak-extensionality property.

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