ASYMMETRIES ON CUT-AND-PROJECT SETS AND RELATED TILINGS
(切断射影集合とそれに関するタイル張りの非対称性)

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Abstract. In order to characterize the (a)symmetries of cut-and-project sets, we prove the following: Given a cut-and-project set with the two projections being injective on the lattice. Then it is fixed by an affine endofunction if and only if (1) the window restricted on the projection of the lattice is fixed by another affine endofunction, and (2) both affine endofunctions induce the same endofunction on the lattice via the two projections. By this theorem, we prove that any Thurston-Akiyama tilings are asymmetric with respect to any affine functions.

1. Introduction

Materials discovered by Shechtmann *et al.* in 1984, unlike ordinary crystals that used to be, were aperiodic and had unusual rotation symmetries. Those materials are now called *quasicrystals*. For the arrangements of atoms in quasicrystals, many mathematical idealizations have been proposed. Among them, we are concerned with *cut-and-project sets*, introduced by de Brujin [4, 5] to discuss the aperiodicity of Penrose tilings. In this paper, we characterize the affine functions that fix a given cut-and-project set, by which we prove that no affine function fixes Thurston-Akiyama tilings. To state our Main Theorem, we recall basic definitions:

Definition 1.1 (Lattice)**.** By a *lattice* of a real vector space $\mathbb{R}^n$, we mean $A\mathbb{Z}^n$ for some $A \in GL_n(\mathbb{R})$.

Following Moody [12], we define a cut-and-project scheme as follows:

Definition 1.2. A *cut-and-project scheme* consists of two components: a direct product $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ of real vector spaces, and a lattice D in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ such that, for the canonical projections $\Pi_1 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ and $\Pi_2 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$,

1. Π_1 is injective on D, and
2. $\Pi_2[D]$ is dense in $\mathbb{R}^{d_2}$.

We simply denote the cut-and-project scheme by $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, D)$. We write $D_i := \Pi_i[D]$ $(i = 1, 2)$. $\mathbb{R}^{d_1}$ is often called the *physical space* and $\mathbb{R}^{d_2}$ the *internal space*. We call D the *generating lattice*.

Definition 1.3 (Window)**.** We say a subset Ω of the internal space $\mathbb{R}^{d_2}$ is a *window* if Ω is a relatively compact set with non-empty interior.

Definition 1.4. A subset C of $\mathbb{R}^{d_1}$ is called a *cut-and-project set* if there is a cut-and-project scheme $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, D)$ and a window Ω of $\mathbb{R}^{d_2}$ such that

$$C = \left\{ \Pi_1(\boldsymbol{x}) : \boldsymbol{x} \in D,\, \Pi_2(\boldsymbol{x}) \in \Omega \right\}. \tag{1}$$

We simply denote it by $\mathbf{CP}(D, \Omega)$.

The cut-and-project sets are instances of what Y. Meyer [11] had already introduced under the name of "model sets", and model sets are instances of *relatively dense harmonious sets* (now called *Meyer sets*), which he introduced as sampling points of the Fourier series of aperiodic functions. Cut-and-project sets and Meyer sets are studied intensively from various points of view (see Moody [13] for mathematical point of view).

We are motivated by the followings works on symmetries of cut-and-project sets:

- K. Niizeki [14, 15, 16, 17] constructed a 2D cut-and-project set from n-th primitive roots of 1, for each $n = 5, 8, 10, 12$ (that is, the value of the Euler function of n is 4). He called them *4D n-gonal quasilattices*. He discussed the associated 2D tilings (we call them *Niizeki tilings*), the self-similarity and inflation symmetries.
- P. A. B. Pleasants [20] studied sufficient conditions on the lattice, projections and window in the cut-and-project construction for various properties of resulting cut-and-project set, such as uniformity, diffraction, Ammann bars, symmetries, inflation and local rules. But it seems that he did not discussed necessary conditions for various symmetries of the cut-and-project sets. He also constructed cut-and-project sets from modules over algebraic number fields.

In this paper, we show that the symmetries (periodicities, rotation symmetries and inflation symmetries) of cut-and-project sets correspond to the symmetries of the windows. To be precise, our Main Theorem states the following:

> Given a cut-and-project set with the two projections being injective on the lattice. Then it is fixed by an affine endofunction if and only if
> 1. the window restricted on the projection of the lattice is fixed by another affine endofunction, and
> 2. both affine endofunctions induce the same endofunction on the lattice via the projections.

The assumption of the theorem is satisfied by n-gonal quasilattices. Thus we can prove that an affine function that fixes an n-gonal quasilattice is exactly a rotation around the origin of angle $2\pi k/n$ for some k, and can characterize inflation symmetries of the n-gonal quasilattices. Since the n-gonal quasilattices are the vertex sets of Niizeki tilings (see figure 1), we can derive the aperiodicity of the Niizeki tilings.

Our theorem also establishes that any Thurston-Akiyama tilings are asymmetric with respect to any affine functions. They were introduced in W. P. Thurston [22]. He suggested a construction of Thurston-Akiyama tilings from Pisot numbers. Here a Pisot number is, by definition, an algebraic integer greater than 1 where every conjugate other than itself has modulus strictly less than 1. Akiyama [1, 2] was concerned with a property (W) introduced by Hollander [8], and proved that every Pisot number β with the property (W) yields indeed a tiling by Thurston's construction.

Unlike Niizeki tilings, it is not simple to prove that the Thurston-Akiyama tilings are asymmetric with respect to any affine function. It is because the fractal boundaries of the tiles make it difficult to comprehend the (a)symmetries of the tiles (see figure 2). Given a Thurston-Akiyama tiling, from each tile we choose a point that plays central role in the construction of the tiling. We call the resulting set the

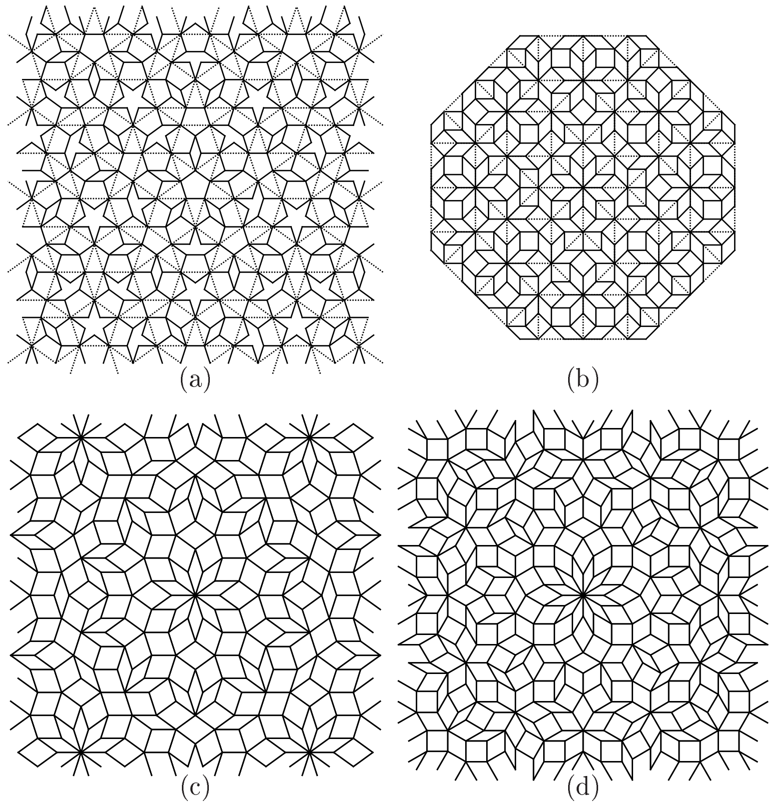

FIGURE 1. Niizeki aperiodic tilings: (a) 5-fold, (b) 8-fold, (c) 10-fold non-Bravais type, and (d) 12-fold. The pictures are courtesy of Prof. Komajiro Niizeki.

characteristic point set of the Thurston-Akiyama tiling. We show that any characteristic point sets become cut-and-project sets. Then by applying our theorem, we finally establish the asymmetries of the tiling.

The paper is organized as follows: In the next section, the basic properties of cut-and-project sets and Main Theorem, which characterizes the (a)symmetries of cut-and-project sets, are presented. Section 3 is preparation to discuss the asymmetries of vertex sets of Niizeki tilings and characteristic point sets of Thurston-Akiyama tilings. We go into details on algebraically constructed cut-and-project schemes. We give two examples of applications of our Main Theorem. First, we discuss the asymmetries of Niizeki tilings with our framework, in Section 4. Second, we prove the asymmetries of Thurston-Akiyama tilings, in Section 5. The definition of the tilings is given in Section 5.1.

2. Asymmetries of Cut-and-Project Sets

Let $B_n(\boldsymbol{x}, r)$ be the open ball of radius $r > 0$ centered around $\boldsymbol{x}$ in $\mathbb{R}^n$.

Definition 2.1. A set C in $\mathbb{R}^n$ is called a *Delaunay set*, if it has the following two properties:

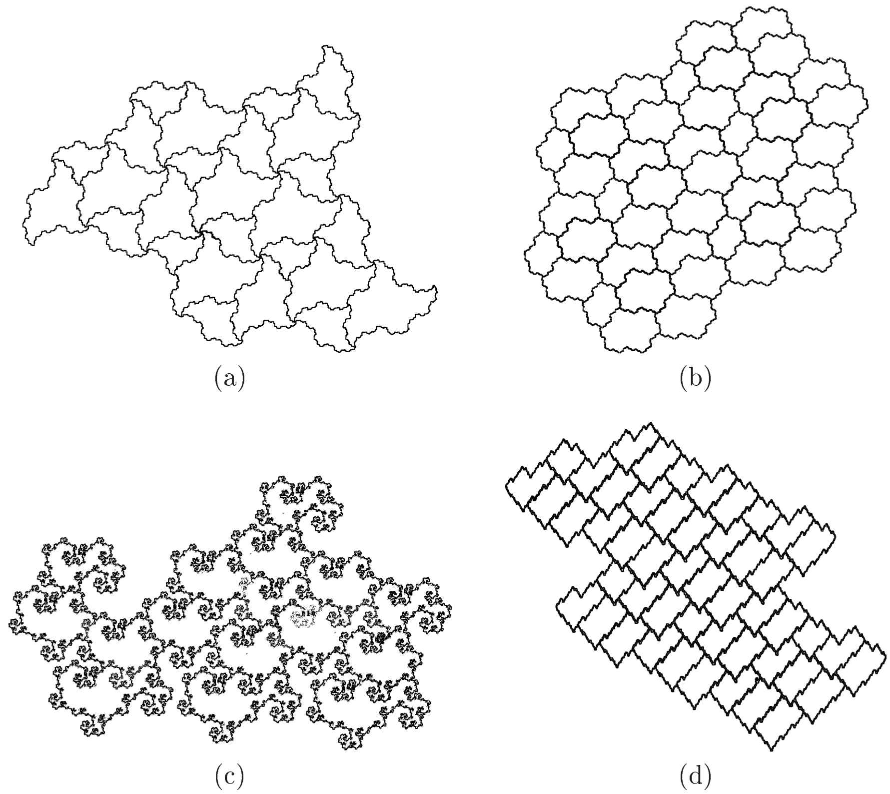

FIGURE 2. Thurston-Akiyama tilings by Pisot numbers whose irreducible polynomials are (a) $x^3 - x - 1$ (minimal Pisot number), (b) $x^3 - x^2 - x - 1$ (Rauzy fractal) (c) $x^3 - 3x^2 + 2x - 1$, and (d) $x^3 - 3x^2 + 1$. They have fractal boundaries.

1. *Uniformly discrete*: There exists a distance $r > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$, $B_n(\boldsymbol{x}, r)$ contains at most one point of C.
2. *Relatively dense*: There exists a distance $R > 0$ such that for all $\boldsymbol{x} \in \mathbb{R}^n$, $B_n(\boldsymbol{x}, R)$ contains at least one point of C.

This concept is named after B. N. Delone (alternative spelling: B. N. Delaunay).

For subsets X and Y of a real vector space, put $X+Y = \{\, \boldsymbol{x}+\boldsymbol{y} : \boldsymbol{x} \in X,\, \boldsymbol{y} \in Y \,\}$ and $X - Y = \{\, \boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in X,\, \boldsymbol{y} \in Y \,\}$. For $y \in Y$, $X + \boldsymbol{y} = \{\, \boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in X \,\}$ and $X - \boldsymbol{y} = \{\, \boldsymbol{x} - \boldsymbol{y} : \boldsymbol{x} \in X \,\}$.

We recall characterizations of *Meyer sets* [11].

Proposition 2.2 (Meyer [11])**.** *For any $\Lambda \subseteq \mathbb{R}^n$, the followings are equivalent:*

1. *Λ is a Delaunay set such that $\Lambda \subseteq C + F$ for some cut-and-project set C and some finite set F.*
2. *Λ is a Delaunay set such that $\Lambda - \Lambda \subseteq \Lambda + F$ for some finite set F (This is the definition of a Meyer set).*
3. *Λ and $\Lambda - \Lambda$ are Delaunay sets. This characterization is due to Lagarias* [9].

As a corollary, the following holds:

Proposition 2.3. *Every cut-and-project set is a Meyer set.*

Given a cut-and-project scheme $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, D)$. Because $\Pi_1|_D : D \to D_1$ is an injection, we can define the natural homomorphism $\flat$ from D_1 to D_2 as follows:

$$\flat = \Pi_2 \circ (\Pi_1|_D)^{-1} : D_1 \to D_2. \tag{2}$$

For $u \in D_1$ and $U \subseteq D_1$, $u^\flat$ and $U^\flat$ stand for $\flat(u)$ and $\flat[U]$.

For a subset X of a real vector space, $\mathrm{Inn}(X)$ and $\mathrm{Cl}(X)$ stand for the interior and the closure of X, and $\langle X \rangle$ stands for the Abelian subgroup generated by X. For x of a real vector space, the norm is denoted by $\|x\|$.

Theorem 2.4. *Suppose a cut-and-project set C is defined by a cut-and-project scheme $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, D)$ and a window Ω such that the canonical projection $\Pi_2 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ is injective on D. Then $\langle C - C \rangle = D_1$.*

Our Main Theorem states the following:

> Given a cut-and-project set with the two canonical projections being injective on the lattice. Then it is fixed by an affine endofunction if and only if
> 1. The window restricted on the projection of the lattice is fixed by another affine endofunction, and
> 2. both affine endofunctions induce the same endofunction on the lattice via the two projections.

To be more precise, Main Theorem states the following:

Theorem 2.5 (Main)**.** *Let C be a cut-and-project set defined by a cut-and-project scheme $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2},\ D)$ such that the canonical projection $\Pi_2 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ is injective on D.*

Given any affine function $f_1 : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$. Then

1. *The followings are equivalent*
   (a) $f_1[C] = C$.
   (b) $f_1[D_1] \subseteq D_1$, *and there exists an affine function $f_2 : \mathbb{R}^{d_2} \to \mathbb{R}^{d_2}$ which satisfies*
      (i) $f_2[D_2] \subseteq D_2$,
      (ii) $\flat \circ f_1|_{D_1} = f_2|_{D_2} \circ \flat$,
      (iii) $f_2[\Omega \cap D_2] = \Omega \cap D_2$.
2. *The first assertion still holds even if we replace the two occurrences of $=$ in* (1a) *and* (1(b)iii) *with $\subseteq$.*

3. Algebraically Constructed Cut-and-project Schemes

The assumption of Main Theorem (Theorem 2.5) are satisfied by *algebraically constructed cut-and-project schemes*, which we here introduce. These schemes have reasonably important examples:

- the n-gonal quasilattices, the vertex sets of Niizeki tilings,
- the "characteristic point set" derived from Thurston-Akiyama tilings.

In later sections, we show the asymmetries of Niizeki tilings and Thurston-Akiyama tilings.

Definition 3.1 (Algebraically Constructed Cut-and-Project Scheme)**.** First we give data for the generating lattice D of an algebraically constructed cut-and-project scheme.

	physical space	internal space
dimension	d_1	d_2
number of related real conjugates	r_1	r_2
number of related non-real conjugates	$2s_1$	$2s_2$

TABLE 1. Data for projections of algebraically constructed cut-and-project scheme.

Let K be an algebraic number field, $d = [K : \mathbb{Q}]$. Put $d = r + 2s$. Let $\sigma_1, \sigma_2, \dots, \sigma_r$ be the associated $\mathbb{Q}$-embeddings from K to $\mathbb{R}$ and $\sigma_{r+1}, \dots, \sigma_{r+2s}$ be the associated $\mathbb{Q}$-embeddings from K to $\mathbb{C}$ where $\sigma_{r+2i}(z) = \overline{\sigma_{r+2i-1}(z)}$ for $1 \le i \le s$. Let $\omega_1, \dots, \omega_d$ be a basis for $K/\mathbb{Q}$. Let V be a matrix:

$$V = \left(\sigma_i(\omega_j)\right)_{1\le i,j\le d} .$$

If $s > 0$, then some element in $V\mathbb{Z}^d$ can have non-real components in the last $2s$ lines. So, as a generating lattice D of $\mathbb{R}^d$, we instead consider

$$D = Q_X V \mathbb{Z}^d. \tag{3}$$

Here the matrix Q_X is defined by

$$Q_X = E_r \oplus \bigoplus_{1\le i\le s} Y_X, \tag{4}$$

where E_r is the identity matrix of size r, $Y_X = XY_{E_2}$, $X \in GL_2(\mathbb{R})$ and

$$Y_{E_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -\sqrt{-1} & \sqrt{-1} \end{pmatrix}.$$

For matrixes M and N, $M \oplus N$ is the matrix of the form $\begin{pmatrix} M & O \\ O & N \end{pmatrix}$.

Second, we give data for projections of the cut-and-project scheme (see also Table 1).

- d_1 is the dimension of the physical space.
- d_2 is the dimension of the internal space.
- r_1 is the number of the real conjugates related to the physical space.
- s_1 is the half of the number of the non-real conjugates related to the physical space. Thus $d_1 = r_1 + 2s_1$.
- r_2 is the number of the real conjugates related to the internal space.
- s_2 is the half of the number of the non-real conjugates related to the internal space. Thus $d_2 = r_2 + 2s_2$.
- Thus $r = r_1 + r_2$ and $s = s_1 + s_2$.
- The projection to the physical space is represented as the $d_1 \times d$ matrix

$$\mathit{\Pi}_1 = \begin{pmatrix} E_{r_1} & O_{r_1,r_2} & O_{r_1,2s_1} & O_{r_1,2s_2} \\ O_{2s_1,r_1} & O_{2s_1,r_2} & E_{2s_1} & O_{2s_1,2s_2} \end{pmatrix}. \tag{5}$$

- The projection to the internal space is represented as $d_2 \times d$ matrix

$$\mathit{\Pi}_2 = \begin{pmatrix} O_{r_2,r_1} & E_{r_2} & O_{r_2,2s_1} & O_{r_2,2s_2} \\ O_{2s_2,r_1} & O_{2s_2,r_2} & O_{2s_2,2s_1} & E_{2s_2} \end{pmatrix}. \tag{6}$$

Then we state the following:

Theorem 3.2. *For $i = 1, 2$, let $\mathit{\Pi}_i$ be a projection from $\mathbb{R}^d$ to $\mathbb{R}^{d_i}$ which is defined above. Then,*

1. *Each $\mathit{\Pi}_i$ is injective on D, and*
2. *D_i is dense in $\mathbb{R}^{d_i}$.*

Thus every algebraically constructed cut-and-project scheme $(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, D)$ is indeed a cut-and-project scheme.

It is convenient to employ *dual basis* (Lang [10, Corollary 1, Chapter VIII, Section 5]).

Proposition 3.3 (Dual Basis)**.** *Let $\omega_1, \ldots, \omega_d$ be a basis for $K/\mathbb{Q}$. Then there exists the dual basis $\omega_1^*, \ldots, \omega_d^*$ of $K/\mathbb{Q}$. It is defined by the properties $\mathrm{Tr}(\omega_i\omega_j^*) = \delta_{ij}$ with δ_{ij} being a Kronecker's delta. Here the trace Tr is defined for any $x \in K$ by $\mathrm{Tr}(x) = \sum_{i=1}^d \sigma_i(x)$. Hence*

$$({}^tV)^{-1} = \big(\sigma_i(\omega_j^*)\big)_{i,j}. \tag{7}$$

4. Asymmetry of n-gonal Quasilattices

Simple examples of algebraically constructed cut-and-project sets are 4D *n-gonal quasilattices* (Niizeki [15, 16, 17, 14]). They are the vertex sets of Niizeki tilings. We discuss the (a)symmetries by Main Theorem (Theorem 2.5). First we recall 4D n-gonal quasilattices.

Let ζ_n be the complex number $\exp(2\pi\sqrt{-1}/n)$ with $n = 5$, 8, 10 or 12 (i.e., $\varphi(n) = 4$ where φ is the Euler function). Put four $\mathbb{Q}$-isomorphisms σ_i ($i = 1, 2, 3, 4$) from $\mathbb{Q}(\zeta_n)$ to itself be

$$\sigma_1(\zeta_n) = \zeta_n, \quad \sigma_2(\zeta_n) = \overline{\zeta_n}, \quad \sigma_3(\zeta_n) = \zeta_n^{i_n} \quad \text{and} \quad \sigma_4(\zeta_n) = \overline{\zeta_n^{i_n}}.$$

Here $i_n(\neq 1, n-1)$ is any positive integer with $(i_n, n) = 1$.

Let Ω_n be a regular n-gon in the Gauss plane with the center being the origin. The *n-gonal quasilattice* is, by definition,

$$Q_n(\Omega_n) = \big\{\, x \in \mathbb{Z}[\zeta_n] : \sigma_3(x) \in \Omega_n \,\big\}.$$

The n-gonal quasilattice is an instance of an algebraically constructed cut-and-project set of Section 3: Set $K = \mathbb{Q}(\zeta_n)$, $d = [K : \mathbb{Q}] = \varphi(n) = 4$. See Table 2. A basis $\omega_1, \ldots, \omega_d$ is

$$1,\ \zeta_n,\ \zeta_n^2,\ \zeta_n^3.$$

$X = E_2/\sqrt{2}$. Then the generating lattice D will be the following:

$$L_n = \begin{pmatrix} 1 & \Re\zeta_n & \Re\zeta_n^2 & \Re\zeta_n^3 \\ 0 & \Im\zeta_n & \Im\zeta_n^2 & \Im\zeta_n^3 \\ 1 & \Re(\sigma_3(\zeta_n)) & \Re(\sigma_3(\zeta_n^2)) & \Re(\sigma_3(\zeta_n^3)) \\ 0 & \Im(\sigma_3(\zeta_n)) & \Im(\sigma_3(\zeta_n^2)) & \Im(\sigma_3(\zeta_n^3)) \end{pmatrix} \mathbb{Z}^4.$$

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, D_1 becomes $\mathbb{Z}[\zeta_n]$, D_2 becomes $\sigma_3\big(\mathbb{Z}[\zeta_n]\big) = \mathbb{Z}\big[\sigma_3(\zeta_n)\big]$, D becomes $\mathbb{Z}[\zeta_n] \times \mathbb{Z}\big[\sigma_3(\zeta_n)\big]$, and $\flat : D_1 \to D_2$ becomes σ_3.

By applying Theorem 3.2, we have the following:

Theorem 4.1. *Let n be 5, 8, 10 or 12. Then $(\mathbb{R}^2 \times \mathbb{R}^2, L_n)$ is an algebraically constructed cut-and-project scheme (see Table 2) where the projections $\mathit{\Pi}_1$ and $\mathit{\Pi}_2$ are defined by*

$$\mathit{\Pi}_1 = (\,E_2 \quad O_{2,2}\,)\,,\ \mathit{\Pi}_2 = (\,O_{2,2} \quad E_2\,) \in M_{2\times 4}(\mathbb{R}).$$

	physical space	internal space
dimension	$d_1 = 2$	$d_2 = 2$
number of related real conjugates	$r_1 = 0$	$r_2 = 0$
number of related non-real conjugates	$2s_1 = 2$	$2s_2 = 2$

TABLE 2. Data for projections of the cut-and-project scheme of n-gonal quasilattices ($n = 5$, 8, 10, 12).

Let $\Omega_n \subseteq \mathbb{R}^2$ be a regular n-gon with the center being the origin. Then $\mathbf{CP}(L_n, \Omega_n)$ is $Q_n(\Omega_n)$ by identifying $\mathbb{C}$ with $\mathbb{R}^2$.

By our Main Theorem, the rotation symmetries and the inflation symmetries of the n-gonal quasilattices are characterized as follows:

Corollary 4.2. *Let n be 5, 8, 10 or 12.*

1. *An affine function f_1 fixes $Q_n(\Omega_n)$, if and only if f_1 represents the rotation around the origin of angle $2\pi k/n$ for some $k = 1, \ldots, n$.*
2. *Let η be greater than 1. Then, η is an inflation symmetry of $Q_n(\Omega_n)$ if and only if $\eta \in \mathbb{Z}\big[2\cos(2\pi/n)\big]$ and $|\eta^*| < 1$ hold. Here η^* means the algebraic conjugate of η defined by $\big(2\cos(2\pi/n)\big)^* = 2\cos(2i_n\pi/n)$.*

The aperiodicities of a Penrose tiling was originally proved by Penrose with the argument of the inflation symmetries and the density of the prototiles [19].

Next Remark gives us some relation between algebras and our Main Theorem.

Remark 4.3. The subset version of Main Theorem (Theorem 2.5) is relevant to an infinite dimensional Lie algebra introduced by Fairie and Zachos [6]. Their Lie algebra is generated from $\{\, J^l_u : l \in \mathbb{N},\ u \in Q_n(\Omega_n) \,\}$ and the relation

$$J^l_u J^m_v = J^{l+m}_{u+v\zeta^l_n} \quad \text{if } u + v\zeta^l_n \in Q_n(\Omega_n).$$

A simple realization is $J^l_u = \exp\big(u\exp(x)\big)\zeta_n^{l\partial_x}$.

By defining $f^l_u : \mathbb{C} \to \mathbb{C}$ as $f^l_u(z) = u + z\zeta^l$, the proviso "$u + v\zeta^l_n \in Q_n(\Omega_n)$" is rephrased as $f^l_u(v) \in Q_n(\Omega_n)$. The condition equivalent to $f^l_u\big[Q_n(\Omega_n)\big] \subseteq Q_n(\Omega_n)$ is given in our Main Theorem.

5. Asymmetry of Thurston-Akiyama Tilings

Thurston introduced tilings based on real number representations with the radixes being Pisot numbers, and studied them with finite state automata [22]. Further, in Akiyama [1, 2], he gave an algebraic characterization of Pisot numbers which yield tilings, and studied the tilings comprehensively. We will establish their tilings are not symmetric with respect to any affine transformations.

In the first Subsection, we recall Thurston-Akiyama tilings are constructed algebraically from Pisot numbers β. For each Thurston-Akiyama tiling, each tile is labeled by Parry's β-expansion. Our key proof idea of asymmetry of the tilings is to assign to each tile a unique point based on the tile's label. We call such point the *characteristic point* of the tile. In Subsection 5.2, to prove the asymmetry of the tiling easy, we will be concerned with an equivalence class $\mathcal{P}$ of tiles by the translation. We will prove that $\mathcal{P}$ corresponds to an asymmetric interval. In Subsection 5.3, we will be concerned with the set $c_\beta[\mathcal{P}]$ of characteristic points of tiles in $\mathcal{P}$. We will prove that $c_\beta[\mathcal{P}]$ is an algebraically constructed cut-and-project set.

The physical space and the internal space are related to each other by the conjugate map of the Pisot number β, and the window is the asymmetric interval from which the asymmetries of $c_\beta[\mathcal{P}]$ will follow through our Main Theorem. Then we will establish the asymmetries of the Thurston-Akiyama tilings.

Below β is a real number greater than 1.

5.1. **Thurston-Akiyama Tilings.** We first recall real number representations with non-integral radixes, and β-expansions of W. Parry [18]. Let $\beta > 1$ be a real number and $\mathcal{A}_\beta := \mathbb{Z} \cap [0, \beta)$. We can represent any real $x \geq 0$ in base β:

$$x = \sum_{i=-m}^{\infty} a_{-i}\beta^{-i} = a_m \cdots a_0.a_{-1}a_{-2}\cdots \quad (a_m, a_{m-1}, \cdots \in \mathcal{A}_\beta). \tag{8}$$

In this case we write

$$x = \rho_\beta(a_m \cdots a_0.a_{-1}a_{-2}\cdots)$$

The sequence $a_m \ldots a_0$ is called the *integer part* and $a_{-1}a_{-2}\cdots$ the *fraction part*. The representation (8) is called *finite* when it ends with the right-infinite repetition 0^ω of 0 (it is often omitted), and is called a *β-expansion* when it satisfies the following *greedy condition*: $\left|x - \sum_{i=-m}^{n} a_{-i}\beta^{-i}\right| < \beta^{-n}$ for all $n \geq -m$.

Let $\mathbf{Fin}(\beta)$ be the set of non-negative reals which has a finite β-expansion. $\mathbf{Fr}$ denotes the set of all fraction parts of $\mathbb{Z}[\beta]_{\geq 0}$:

$$\mathbf{Fr} := \left\{\omega : .\omega \text{ is the } \beta\text{-expansion of some } x \in \mathbb{Z}[\beta]_{\geq 0}\right\}.$$

The β-expansion of 1, denoted by $d(1,\beta)$, is, by definition, a sequence $t_1t_2\cdots$ where $t_1, t_2, \ldots \in \mathcal{A}_\beta$ and $r_0 := 1$, $t_i := \lfloor \beta r_{i-1} \rfloor$ and $r_i := \{\beta r_{i-1}\}$ for all $i \geq 1$. Here for a given $x \geq 0$, $\lfloor x \rfloor$ is the greatest integer lesser than or equal to x, and $\{x\}$ is $x - \lfloor x \rfloor$. The concatenation of sequences p and q is denoted by pq. Define $d^*(1,\beta)$ by

$$d^*(1,\beta) := \begin{cases} d(1,\beta) & \text{if } d(1,\beta) \text{ is an infinite sequence,} \\ (t_1 \cdots t_{n-1}(t_n - 1))^\omega & \text{if } d(1,\beta) = t_1 \cdots t_n. \end{cases}$$

Here $(t_1 \cdots t_{n-1}(t_n - 1))^\omega$ is the right-infinite repetition of the finite sequence $t_1 \cdots t_{n-1}(t_n - 1)$. According to Schmidt [21], for every Pisot number β, $d^*(1,\beta)$ is *eventually periodic*. That is, the following holds:

Proposition 5.1. *If β is a Pisot number, then $d^*(1,\beta) = t_1 \cdots t_n(t_{n+1} \cdots t_{n+p})^\omega$ for some $n + p \geq 1$ and some $t_1, \ \ldots, \ t_{n+p} \in \mathcal{A}_\beta$.*

For two infinite sequences $p = p_1p_2\cdots$ and $q = q_1q_2\cdots$, we write $p <_{\text{lex}} q$, if there exists some $i \geq 1$ such that for all $k < i$, $p_k = q_k$ and $p_i < q_i$. The lexicographical order between β-expansions characterizes the numerical order between them:

Proposition 5.2. *For all β-expansions $.\omega$ and $.\eta$, we have $\omega <_{\text{lex}} \eta$ if and only if $\rho_\beta(.\omega) < \rho_\beta(.\eta)$.*

A bi-infinite sequence $\cdots a_{-2}a_{-1}a_0a_1a_2\cdots$ $(a_i \in \mathcal{A}_\beta)$ is called *β-admissible* if

$$a_ia_{i+1}\cdots <_{\text{lex}} d^*(1,\beta) \quad (\forall i).$$

Define the β-admissibility of an infinite sequence similarly. Below we will implicitly use W. Parry's result [18]:

Proposition 5.3. *The representation* (8) *is β-expansion if and only if it is β-admissible.*

Definition 5.4. 1. Let $\beta > 1$ be a *Pisot unit*, that is, an algebraic unit which is a Pisot number.
2. Let $K = \mathbb{Q}(\beta)$ be an algebraic number field, $d = [K : \mathbb{Q}]$. As a basis for $K/\mathbb{Q}$, choose $1, \beta, \ldots, \beta^{d-1}$. Put $d = r + 2s$. Let $\sigma_1, \sigma_2, \ldots, \sigma_r$ be the associated $\mathbb{Q}$-embedding from K to $\mathbb{R}$ and $\sigma_{r+1}, \ldots, \sigma_{r+2s}$ be the associated $\mathbb{Q}$-embedding from K to $\mathbb{C}$ where $\sigma_{r+2i}(z) = \overline{\sigma_{r+2i-1}(z)}$ for $1 \leq i \leq s$. Let $\sigma_r = \mathrm{id}$.
3. For every $a \in \mathbb{Z}[\beta]$, put

$$\Phi(a) = {}^t\big(\sigma_1(a), \ldots, \sigma_{r-1}(a), \Re\sigma_{r+1}(a), \Im\sigma_{r+1}(a), \ldots, \Re\sigma_{r+s}(a), \Im\sigma_{r+s}(a)\big). \quad (9)$$

Given a possibly bi-infinite sequence $\cdots t_2t_1t_0.t_{-1}t_{-2}\cdots$ with each t_i being in $\mathcal{A}_\beta$. If the point $\sum_i t_i\Phi(\beta^i) \in \mathbb{R}^{d-1}$ is defined, then it is denoted by $\overline{\rho}_\beta(\cdots t_1t_0.t_{-1}\cdots)$. For each β-expansion $w.\omega$, define the compact set $T_{w.\omega} \subseteq \mathbb{R}^{d_1}$ by

$$S_{w.\omega} := \big\{\, \rho_\beta(vw.\omega) \in \mathbb{Z}[\beta]_{\geq 0} : v \text{ is a finite sequence s.t. } vw.\omega \text{ is a } \beta\text{-expansion} \,\big\},$$
$$T_{w.\omega} := \mathrm{Cl}\big(\Phi[S_{w.\omega}]\big).$$

The compactness of $T_{w.\omega}$ follows from the property of Pisot numbers. We call $T_{.\omega}$ a *tile*. There is a linear map $G_K : \mathbb{R}^{d_1} \to \mathbb{R}^{d_1}$ which commutes the following diagram:

$$\begin{tikzcd} \mathbb{Q}(\beta) \arrow[r, "\times\beta^{-K}"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Q}(\beta) \arrow[d, "\Phi"] \\ \mathbb{R}^{d_1} \arrow[r, "G_K"'] & \mathbb{R}^{d_1} \end{tikzcd}$$

Each tile $T_{.\omega}$ satisfies the following *inflation subdivision principle*:

$$G_1(T_{.\omega}) = \bigcup_a T_{.a\omega},$$

where the union runs over all $a \in \mathcal{A}_\beta$ such that $.a\omega$ is a β-expansion.

Assume that β has the following property:

$$(\mathrm{W}) : \forall x \in \mathbb{Z}[\beta^{-1}]_{\geq 0},\ \forall \varepsilon > 0,\ \exists y, z \in \mathbf{Fin}(\beta) \text{ such that } |z| < \varepsilon \text{ and } x = y - z.$$

Each $d \in \mathbb{Z}_{>1}$ has infinitely many such Pisot number β. Actually, according to [7], if a polynomial $X^d - a_1X^{d-1} - a_2X^{d-2} \cdots - a_{d-1}X - 1 \in \mathbb{Z}[X]$ satisfies $a_1 \geq a_2 \geq \cdots \geq a_{d-1} \geq 1$, then it has a zero point $\beta > 1$ such that β is a Pisot number and (F) : $\mathbb{Z}_{\geq 0}[\beta^{-1}] \subseteq \mathbf{Fin}(\beta)$. Then, by [2, Proposition 3], β satisfies the property (W).

Given a tile $T_{.\omega}$, an inner point x of it is called *exclusive* if x is not in another tile $T_{.\eta}$.

Remark 5.5. Let $v\omega$ and $\pi\eta$ be β-admissible. If $\overline{\rho}_\beta(\pi.\eta) \in \mathrm{Inn}(T_{v.\omega})$ then v is a suffix of π and $\eta = \omega$.

According to Akiyama [2, Proposition 2],

Proposition 5.6. *The property* (W) *holds if and only if each tile $T_{.\omega}$ has an exclusive inner point.*

Here after we assume β satisfies (W) and is a Pisot unit. The set

$$\mathcal{T}_\beta := \{\, T_{.\omega} : \omega \in \mathbf{Fr} \,\}$$

provides a self-affine tiling of $\mathbb{R}^{d_1}$, and we call it a *Thurston-Akiyama tiling.*

5.2. **Prototiles.** We will be concerned with an equivalence class $\mathcal{P}$ of tiles under the translation. We call it a *prototile*. Each tile has a β-expansion as the label where β is a Pisot unit that satisfies (W). Because of (W) and Proposition 5.6, by the label of a tile we can tell which prototile the tile belongs to. Because of Proposition 5.2, we can prove that the set of β-expansions of the tiles in $\mathcal{P}$ corresponds to a left-closed-right-open interval.

Definition 5.7. Define an equivalence relation $\equiv\, \subseteq \mathcal{T}_\beta \times \mathcal{T}_\beta$ by

$$T \equiv T' \iff \text{for some } \boldsymbol{x} \in \mathbb{R}^{d_1},\, T = T' + \boldsymbol{x}.$$

We call an element in the factor set $\mathcal{T}_\beta/\!\equiv$ a *prototile*.

Definition 5.8. Let β be a Pisot unit with the property (W). Then, for $d^*(1,\beta) = t_1 t_2 \cdots$, we have a finite set $\{\, t_i t_{i+1} \cdots : i \geq 1 \,\}$ of infinite sequences, because of Proposition 5.1. Let the set be enumerated as follows

$$d^*(1,\beta) = p_1 >_{\mathrm{lex}} p_2 >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} p_N,$$

For the convenience, we define p_{N+1} by 0^ω. For $\omega \in \mathbf{Fr}$, define

$$[\omega] := \min\{\, p_k : p_k >_{\mathrm{lex}} \omega,\ N \geq k \geq 1 \,\}$$

where min is the minimum with respect to the lexicographical order $>_{\mathrm{lex}}$.

Lemma 5.9 (cf. Akiyama [2, Lemma 5])**.** *For all ω and η in $\mathbf{Fr}$,*

1. $[\omega] \geq_{\mathrm{lex}} [\eta] \iff T_{.\omega} - \overline{\rho}_\beta(.\omega) \subseteq T_{.\eta} - \overline{\rho}_\beta(.\eta)$,
2. $[\omega] >_{\mathrm{lex}} [\eta] \Longrightarrow \mathrm{vol}(T_{.\omega}) < \mathrm{vol}(T_{.\eta})$.

Remark 5.10. Because of Lemma 5.9, the equivalence relation $\equiv$ defined in Definition 5.7 is actually a congruence relation.

Lemma 5.11. *Suppose that ω is β-admissible and $1 \leq k \leq N$. Then*

$$p_{k+1} \leq_{\mathrm{lex}} \omega \iff \rho_\beta(.p_{k+1}) \leq \rho_\beta(.\omega).$$

Combined with Lemma 5.9, we have the following:

Theorem 5.12. *For every prototile $\mathcal{P} \in \mathcal{T}_\beta/\!\equiv$ there exists $k \in \{\, 1, \ldots, N \,\}$ such that $\mathcal{P} = \big\{\, T_{.\omega} : \omega \in \mathbf{Fr},\, \rho_\beta(.\omega) \in \big[\rho_\beta(.p_{k+1}),\, \rho_\beta(.p_k)\big) \,\big\}$.*

5.3. **Cut-and-Project Set of Characteristic Points.** We will prove that the set $c_\beta[\mathcal{P}]$ of characteristic points of tiles each in prototile $\mathcal{P}$ is an algebraically constructed cut-and-project set. The physical space and the internal space are related to each other by the conjugate map Φ of the Pisot number β. The window is the left-closed-right-open interval given in Subsection 5.2. The only affine function that fixes the interval is identity. By our Main Theorem, it is also the case for the cut-and-project set $c_\beta[\mathcal{P}]$. Because the tiles in the prototile $\mathcal{P}$ are of the same form, we can easily relate to the asymmetries of $c_\beta[\mathcal{P}]$ to the asymmetry of the Thurston-Akiyama tiling itself, by using the compactness of the tiles.

	physical space	internal space
dimension	$d_1 = d - 1$	$d_2 = 1$
number of related real conjugates	$r_1 = r - 1$	$r_2 = 1$
number of related non-real conjugates	$2s_1 = 2s$	$s_2 = 0$

TABLE 3. Data for projections of the cut-and-project scheme for characteristic point set of Thurston-Akiyama tilings.

Definition 5.13. For each $\omega \in \mathbf{Fr}$, we call $\overline{\rho}_\beta(.\omega)$ the *characteristic point* of the tile $T_{.\omega}$.

Because of Proposition 5.6, $T_{.\omega} = T_{.\eta}$ implies $.\omega = .\eta$. So we can define an injection

$$c_\beta : \mathcal{T}_\beta \to \Phi\big[\mathbb{Z}[\beta]\big]$$

such that for each tile $T \in \mathcal{T}_\beta$, $c_\beta(T)$ is the characteristic point of T.

Lemma 5.14. *Following Definition 3.1, data of Definition 5.4(2) give rise to an algebraic cut-and-project scheme*

$$\big(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2},\, Q_{E_2/\sqrt{2}} V \mathbb{Z}^{d_1+d_2}\big)$$

where the projections Π_1 and Π_2 are defined by Table 3. Then we have

$$\begin{tikzcd}
 & D = Q_{E_2/\sqrt{2}} V \mathbb{Z}^d \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[d, phantom, "\circlearrowleft"] & \\
\Phi\big(\mathbb{Z}[\beta]\big) = D_1 \arrow[rr, shift left, "\flat"] & & D_2 = \mathbb{Z}[\beta] \arrow[ll, shift left, "\flat^{-1} = \Phi"]
\end{tikzcd}$$

Theorem 5.15. *Let β be a Pisot unit that satisfies* (W) *and* $\deg \beta \geq 3$ *following the notation of Definition 5.4. Let $\mathcal{P} \in \mathcal{T}_\beta/\equiv$. Then*

1. *there exists $k = 1, \ldots, N$ such that the set $c_\beta[\mathcal{P}]$ of characteristic points of the tiles in $\mathcal{P}$ is*
$$c_\beta[\mathcal{P}] = \mathbf{CP}\Big(Q_{E_2/\sqrt{2}} V \mathbb{Z}^d,\ \big[\,\rho_\beta(.p_{k+1}),\, \rho_\beta(.p_k)\,\big)\Big)$$
where we define $p_{N+1} := 0^\omega$;
2. *if an affine function fixes $c_\beta[\mathcal{P}]$ then it is the identity; and*
3. *$c_\beta[\mathcal{P}]$ has no inflation symmetry.*

Remark 5.16. A *β-integer* is, by definition, a real number $\rho_\beta(p.)$ for some β-admissible p . According to [3], the set of all β-integers is a Meyer set if $d^*(1, \beta)$ is eventually periodic. On the other hand, the "set of all β-fraction"

$$\begin{aligned}
c_\beta[\mathcal{T}_\beta] &= \big\{\, \overline{\rho}_\beta(.p) : p \text{ is } \beta\text{-admissible and in } \mathbb{Z}[\beta] \,\big\} \\
&= \bigcup_{1 \leq i \leq N} \mathbf{CP}\Big(Q_{E_2/\sqrt{2}} V \mathbb{Z}^d,\ \big[\,\rho_\beta(.p_{i+1}),\, \rho_\beta(.p_i)\,\big)\Big) \\
&= \mathbf{CP}\big(Q_{E_2/\sqrt{2}} V \mathbb{Z}^d,\, [0, 1)\big)
\end{aligned}$$

is a cut-and-project set, which is a Meyer set.

Here is our Main Result:

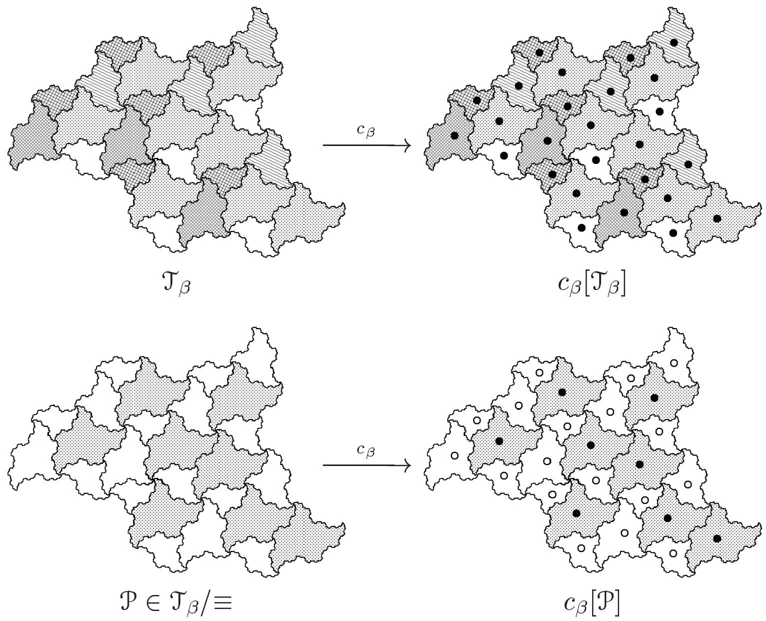

FIGURE 3. The tiling $\mathcal{T}_\beta$ generated by the minimal Pisot number β, a prototile $\mathcal{P} \in \mathcal{T}_\beta/\equiv$, and their characteristic point sets.

Theorem 5.17 (Asymmetry of the Thurston-Akiyama Tilings)**.** *Let $F : \mathbb{R}^{\deg \beta - 1} \to \mathbb{R}^{\deg \beta - 1}$ be an affine function with $\{ F[T] : T \in \mathcal{T}_\beta \} = \mathcal{T}_\beta$. Then F is the identity.*

After the authors completed the proof, they are informed by Dr. Sadahiro that for the Thurston-Akiyama tiling of the minimal Pisot number, the set of points belonging to three different tiles is a cut-and-project set. So by applying our Main Theorem to the set, we can directly establish the asymmetry of the Thurston-Akiyama tiling of the minimal Pisot number.

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