SEARCH OF MATHEMATICAL STRUCTURES USING A COMPUTER

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1. Introduction

There are many reserches of mathematical structures using computers. And some difficulties arise in carrying on the task. The following talk consists of three parts.

Part 1 discusses the nature of those difficulties. D1: Limit of a speed of computations, D2: An abundance of results, D3: Problem of all and D4: Difficulties of a global consideration.

In Part 2, we explain problems which relates these difficulties. P1: Construction of covering systems, P2: Rational Beatty sequences, P3: The linear diophantine problem of Frobenius and P4: RDS Problem.

In Part 3, we propose some devices to overcome the difficulties. M1: Make use of diagrams, M2: Trunk and Star Principle, M3: Make $N \to Q$ and Cell Principle, M4: Semi-local.

2. Difficulties

We take up the following four difficulties D1 - D4.

D1: Limit of a speed of computations

In 1981, Bohman and Fröberg [1] gave mathmatical evidence indicating the truth of the conjecture $G(3) = 4$, i.e. that every sufficiently large numbers can be written as the sum of at most 4 positive cubes.

For the purpose, they tried to determine the density of numbers requiring 5 cubes up to $4 \cdot 10^{11}$. But they could not investigate all numbers $\leq 4 \cdot 10^{11}$. Instead they searched samples, for example the numbers contained in $[4 \cdot 10^{11},\ 4 \cdot 10^{11} + 10^{6}]$.
Note that Western [24] conjectured that these numbers are contained in $[1, 10^{14}]$.

D2: An abundance of results

In many cases, a computer supplies us with numerous computational results. To obtain a structure theory, it is necessary to arrange suitably this flood of numbers.
On the other hand, for example, the denominators and numerators of rational points of an elliptic curve increase with rapidity. And it is difficult to see through the arithmetical nature of these large numbers.

D3: Problem of All

In 1966, Lander, Parkin and Selfridge [5] conducted a series of computer searches of the following equations.

$$\sum_{i=1}^{m} {x_i}^k = \sum_{j=1}^{n} {y_j}^k \quad (1 \leq m \leq n), \quad x_i, y_j \in N. \tag{1}$$

In particular, they found 10 non-trivial solutions of

$$n = x^6 + y^6 + z^6 = u^6 + v^6 + w^6, \tag{2}$$

for smaller n's. English mathematicians observed a curious property (PA).

(PA) The 9 of these 10 solutions of (2) satisfy also the relation

$$x^2 + y^2 + z^2 = u^2 + v^2 + w^2. \tag{3}$$

But one solution $(x, y, z; u, v, w) = (25, 62, 138; 82, 92, 135)$ does not satisfy (3). It seems the reason of (PA) is yet in the dark. On the other hand, many attempts have been done to obtain parametric solutions which satisfy simultaneously two relations (2) and (3). It is notable that these attempts seem to lack in constructing *all* solutions.

Not only for constructing problems, the importance of all property is obvious. We refer to this problem in the following as "Problem of All".

D4: Difficulties of a global consideration

A computer works if it is given a concrete algorithm. But it is weak in treating a global problem. For example, various games such as "Othello", "Chess", "Shōgi" and "Go-game" are treated by a computer [6].

About the former two, a computer defeats even an expart. And recent shōgi softs have reached the level of Top-amateur. On the other hand, Go-game softs are not improved for the reason of the globality of the game. In mathematics, some extremal problems can be solved applying a *Greedy Algorithm*. But for complicated extremal problems, global considerations are needed. And this is a weak point of a computer.

3. Problems

The only way to reach a theory in mathematics is to try to solve some appropriate problems. Thus we explain problems which relate to our theme. The problems are that of treated by the author in the past or present. And they have a common taste. To explain it, we introduce the following two concepts.

(N-system) A set of natural numbers which satisfy some mathematical conditions is called a *N-system*. And a relation whose coefficients are in N is called a *N-relation*.

(Way of representations) The following problems P1 - P4 treat N-systems which satisfy some conditions with respect to ways of certain reresentations. In paticular, the concerns are focused on

1. the set whose numbers are *not representable* in a certain way,
2. the numbers which have exactly two expressions,
3. the interaction of different kinds of representations. For example, (PA) stated in D2 is of this type.

N.B. Since many of the papers [7] - [22] are difficult to obtain. Any interested reader is recommended to access "rmorikawa@muj.biglobe.ne.jp". Then the offprints of the papers will be sent.

Notations: (a, b) means GCD of a and b. $\bar{N} = N \cup \{0\}$. For a tree $T = (V; E)$, V means the set of vertices of T, and E the set of arcs. We usually denote $v \in T$ for $v \in V$.

P1. Costruction of a covering system

A system of pairs $\{(r_i, m_i)|\ 1 \le i \le k\}$ is called a *covering system* if it satisfies

1. $m_i \in N, \quad 0 \le r_i \le m_i - 1 \quad (1 \le i \le k)$.
2. For any $n \in N$, there exists suitable i for which $n \equiv r_i \pmod{m_i}$.
3. $m_i \quad (1 \le i \le k)$ are distinct.

Usually m_i's are taken from the divisors of a fixed $M \in N$. The following example with $M = 12$ was given by Darvenport.

$(0, 2), \quad (1, 4), \quad (0, 3), \quad (1, 6), \quad (11, 12)$.

The difficulty in constructing complicated coverings is of type D2.

A. For $M = p_1^{e_1} \cdots p_s^{e_s}$ the cardinality of the choices of m's is $\prod_{i=1}^{s}(e_i + 1) - 1$.

B. We must determine a suitable r for each m.

The devices to overcome these difficulties are given in [8]. We introduced there a method which is called a *P-process* to avoid A. And to overcome B, we defined a tree. (This tree will be explained in T1.)

As for a covering system, there remains an open cojecture due to P. Erdös.

Question: Does there exist a covering system whose moduli m_i $(1 \le i \le k)$ consist only of odd numbers ?

We call this set a *odd covering* . The existence of it looks plausible. And a good use of a computer will be helpful (D2).

P2. Rational Beatty sequences

Take $q,\ a \in N$ with $(q, a) = 1$, and $b \in Z$. We put $S(q,\ a,\ b) = \{[(qn + b)/a]|\ n \in Z\}$ where $[x]$ means the greatest integer $\le x$. We call $S(q, a, b)$ a *rational Beatty sequence.*

Fundamental problems concerning a family of rational Beatty sequences $\{\ S(\ q_i, a_i, b_i)|\ 1 \le i \le k\}$ are the following.

A. Disjoint system: Find a criterion for $\{(q_i,\ a_i)|\ 1 \le i \le k\}$ so that the sequences $S(q_i, a_i, b_i) \quad (1 \le i \le k)$ are made mutually disjoint by taking suitable b's.

B. ECF: A family $\{\ S(\ q_i, a_i, b_i)|\ 1 \le i \le k\}$ is called an *eventual covering family (ECF)* if

$S(q_i, a_i, b_i)$ $(1 \le i \le k)$ are mutually disjoint and the union of them $= Z$.

The problem is to determine the structure of an ECF.

These two problems are treated in [9] - [21]. Here we note the following two remarks.

(a) Let $S_i = S(q_i, a_i, b_i)$ $(i = 1, 2)$ with $(q_i,\ a_i) = 1$. We introduce the numbers

$$q = (q_1,\ q_2), \quad a = (a_1,\ a_2) \quad \text{and} \quad a_i = au_i\ (i = 1, 2).$$

Theorem ([9] or [15]). *Two sets S_1 and $S_2$are disjoint with suitable two integers b_1 and b_2 if and only if*

$$xu_1 + yu_2 = q - 2u_1u_2(a - l) \tag{4}$$

holds with some $(x, y) \in N^2$.

This Theorem means that the weak condition $S_1 \cap S_2 = \emptyset$ implies the strong N-relation (4).

For a disjoint system $\{\ S(\ q_i, a_i, b_i)|\ 1 \le i \le k\}$ with $k \ge 3$, we treat it under the following condition.

$$q = q_i\ (1 \le i \le k) \quad \text{and} \quad (a_i,\ a_j) = 1 \quad \text{for} \quad 1 \le i < j \le k.$$

Let $k = 4$, then (4) gives six N-relations of the form

$$x_{ij}a_i + y_{ij}a_j = q \quad \text{with} \quad (x_{ij},\ y_{ij}) \in N^2,\ 1 \le i < j \le 4.$$

If a_i's satisfy $a_ia_j > q$ for $1 \le i < j \le 4$, these relations are of very tight ones. And there is a *keen* theory [21].

(b) For an ECF, we investigate the structure introducing a correspondence between an ECF and trees. A *standard* ECF corresponds to a pair of trees [14].

Note that there remains an open conjecture.

Fraenkel's conjecture: Let $\{\ S(\ q_i, a_i, b_i)|\ 1 \le i \le k\}$ be an ECF which satisfies the following condition.

$$k \ge 3, \quad \text{and} \quad q_i/a_i\ (1 \le i \le k) \quad \text{are all distinct.}$$

Then we have

$$\{q_i/a_i \mid 1 \le i \le k\} = \{(2^k - 1)/2^m \mid 0 \le m \le k - 1\}. \tag{5}$$

We call ECF of type (5) an *F(k)-Family.*

P3. The linear diophantine problem of Frobenius

Problem: Let $k \in N$ and take $a_1, a_2, \dots a_k \in N$ whose GCD $= 1$. The problem is to determine the following F.

$$F = \text{Max of } n \text{ which is not expressible by } n = \sum_{i=1}^{k} a_ix_i \text{ with all } x_i \in N.$$

This problem is known to be a difficult one. But there is a considerably good theory for $k = 3$ (cf. [22] and the papers cited there). For the case $k = 4$, the problem turns out to be *much* complicated. The author obtained recently a very long and very rough framework for the case. There remains many holes. But the concepts, tools and devices used there are, we think, available for solving structure problems concerning various N-systems. Thus we explain in this talk some key points of the theory.

(A) The first step is to determine the following L. We take four natural numbers (a, b, c, d) for which

$$\text{For any three of } (a, b, c, d), \text{ the GCD of them } = 1, \quad \text{and} \quad b, c, d \ge 2.$$

We put $W = \{(x, y, z, w) \in Z^4 \mid ax + by + cz + dw = 0\}$. And we take four vectors $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \in W$ as follows:
$\boldsymbol{a} = (-a_1, a_2, a_3, a_4),\ \boldsymbol{b} = (b_1, -b_2, b_3, b_4),\ \boldsymbol{c} = (c_1, c_2, -c_3, c_4),\ \boldsymbol{d} = (d_1, d_2, d_3, -d_4).$
where $a_i, b_j, c_m, d_n\ (1 \le i, j, m, n \le 4)$ are all in $\bar{N}$. And a_1, b_2, c_3, d_4 are the smallest positive integers under these signature condition.

Those $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ are not unique for some (a, b, c, d), but a_1, b_2, c_3, d_4 are uniquely determined. We define

$$L = \{(x, y, z, w) \in W \mid |x| < a_1, |y| < b_2, |z| < c_3, |w| < d_4\}.$$

The role of L for the problem is essential. In case $k = 3$,we have $L = \emptyset$. And this fact allows a simple theory. But to determine L for $k = 4$, we must go along a very long and winding road.

(B) Next we make a set

$$\tilde{L} = \{\boldsymbol{v} = (v_1, v_2, v_3, v_4) \in L \mid v_1 > 0\}, \quad \text{and} \quad M = \tilde{L} \cup \{-\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}.$$

For this M, it is almost certain that the following Lemma holds.

(N-basis Lemma) The set M is either of the following two types.

(type 1) All elements of $\boldsymbol{v} \in M$ are represented by

$$\boldsymbol{v} = s\boldsymbol{x} + t\boldsymbol{y} + u\boldsymbol{z} \quad \text{with} \quad s, t, u \in \bar{N} \tag{6}$$

choosing suitable $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in M^3$.

(type 2) There exists $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) \in M^4$ which satisfies the following two conditions:

(a) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{z} + \boldsymbol{w}$

(b) All vectors $\boldsymbol{v} \in M$ with $\boldsymbol{v} \neq \boldsymbol{w}$ is represented by (6).

This Lemma is fundamental. But we cannot yet prove it for all (a, b, c, d).

P4. RDS Problem

The problem "G(3) = 4 ?" is a grand problem. It is notable that Western [24] introduced a bold idea to overcome D1. Namely he obtained the plausible limit 10^{14} for the numbers which need 5 cubes, long before the advent of automatic computers. Since $G(3)$-Problem seems to be too difficult to attack for the present, we try the following RDS Problem, which is named after Ramanujan, Duke and Schulze-Pillot.

We introduce the following notation: Let $k \in N$ and take $a_1, a_2, \ldots, a_k \in N$. We put $f = \sum_{i=1}^{k} a_i {x_i}^2$. We define

$$V(f) = \{\ f(\boldsymbol{x}) \mid \boldsymbol{x} \in \bar{N}^k\}, \quad \text{and} \quad W(f) = \bar{N} - V(f). \tag{7}$$

In 1917, Ramanujan [23] determined the set $(a, b, c, d) \in N^4$ for which $V(\ ax^2 + by^2 + cz^2 + du^2\ ) = \bar{N}$. In a footnote of [23], he remarked

$$W(\ x^2 + y^2 + 10z^2\ ) = \{4^\lambda(16\mu + 6) |\ \lambda, \mu \in \bar{N}\} \cup \{\text{odd numbers}\}.$$

We denote these odd numbers by $\tilde{W}$. He found that $\tilde{W} = \{3, 7, 21, \ldots, 307, 391, \ldots\}$ and noted "this set does not seem to obey any simple law".

In 1990, Duke and Schulze-Pillot proved that $|\tilde{W}| < \infty$ (cf.[3]). But it is notable that their proof does not yield an explicit bound of Max $\tilde{W}$ (D3).

In [23], Ramanujan studied $W(ax^2 + by^2 + cz^2)$ for several $(a, b, c) \in N^3$. His remark is " it appears that in most cases there are no simple law". But we think that the structure of W is a good example of " Trunk and Star Principle". We discuss this problem later in M2.

4. Some devices to overcome the difficulties

M1. Make use of diagrams

To overcome D2, it is useful in many cases to define a suitable correspondence $\varphi : S \longleftrightarrow T$ between a N-system S and a tree $T = (V; E)$. Here we define a correspondence between $S \ni s \longleftrightarrow v \in V$, which we denote $\varphi(s) = v$. And we attach to E *various kinds of* operations, properties, algoithms etc. We explain some examples.

T1. For a covering system $\{(r_i, m_i)|\ 1 \leq i \leq k\}$, we define a tree $T = (V; E)$. For each *ending* vertex, we attach one of m_i's. Let $\varphi(m) = v$. For each arc which appears in the route to v from the source, we attach a congruence $\equiv t \pmod u$ so that "$\equiv r \pmod m$ of (r, m) is the product of all the congruences which appear in the route".

T2. In treating P2, we define a tree (type 1), or a pair of trees (type 2) for a given ECF [14]. We attach a q_i/a_i to each ending vertex. By a similar method used in T1, b_i is determined according to the place of the vertex.

T3. To consider M stated in P3, we assume the validity of N-basis Lemma. Here we need a *generalized* tree, which has three sources and some loops. We attach the N-basis $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ to three sources. And we construct T inductively.

Take $\boldsymbol{u} \in M$ and assume $\varphi(\boldsymbol{u}) = v \in T$. We consider the following three vectors:

$$\boldsymbol{u} + \boldsymbol{x}, \quad \boldsymbol{u} + \boldsymbol{y}, \quad \boldsymbol{u} + \boldsymbol{z}.$$

Now in case $\boldsymbol{u}+\boldsymbol{x}$ is contained in M, we draw an arc from $v \to \tilde{v}$. Here we put $\varphi(\boldsymbol{u}+\boldsymbol{x}) = \tilde{v}$ and consider the operation $+\boldsymbol{x}$ is attached to the arc. Doing the same for $\boldsymbol{u} + \boldsymbol{y}$ and $\boldsymbol{u} + \boldsymbol{z}$, we get T inductively.

T4. We let remind the beautiful Theory of *Dynkin Diagrams*, which are used to classify all the simple Lie groups. The Diagrams consist of $A_m (m \geq 1)$, $B_m (m \geq 2)$, $C_m (m \geq 3)$, $D_m (m \geq 4)$, $E_m (m = 6, 7, 8)$, F_4 and G_2, where m is the cardinality of vertices. For the shapes of them, we refer to Helgason's Textbook [4], p. 477.

<Surgeries on Diagrams >

Let assume that a correspondence $\varphi : S \longleftrightarrow T$ is given. We define a certain surgery $\star$ on T. We denote $T^\star$ the tree obtained by $\star : T \to T^\star$. And investigate the $S^\star$ which corresponds to $T^\star$. A surgery is operated on some of vertices or arcs. And the place in T of the operation is important. We call it a $\star$-*place*. We give some examples.

S1. As for Dynkin Diagrams, we consider A_m as a standard one. And three kinds of surgeries appear.

(a) Make an arc of A_m to $\Longrightarrow$. Then we obtain $B_m (m \geq 2)$, $C_m (m \geq 3)$ and F_4, according to the $\star$-place.

(b) Make the arc of A_2 to $\Rrightarrow$, we obtain G_2.

(c) Making a ramified arc, we get $D_m (m \geq 3)$ or $E_m (m = 6, 7, 8)$. As for E_m, the $\star$-place plays an important role.

S2. As explained in T2, a correspondence between an ECF of type 1 and a tree is defined. In this framework, a surgery "to delete some of ending vertices from a standard one" is defined. And $F(k)$-family corresponds to trees of this type (Proposition 3 of [11]).

S3. In treating P3,we meet a surgery of another kind. As noted in T3, each arc has one of the operations of $F = \{+\boldsymbol{x}, +\boldsymbol{y}, +\boldsymbol{z}\}$. Our concern is the order of appearance of operations along a route of T. Take a vertex $v \in T$. And assume that the operations $+\boldsymbol{y}$, $+\boldsymbol{x}$, $+\boldsymbol{z}$ appear in this order along a route starting from v. We call $< +\boldsymbol{y}, +\boldsymbol{x}, +\boldsymbol{z} >$ an *order segment* of the above route. Using this terminology, we see that, for certain (a, b, c, d),the following property (PB) holds.

(PB) Assume that the order segment $< +\boldsymbol{z},\ +\boldsymbol{y} >$ appears in some route of T. Then the next operation must be $+\boldsymbol{x}$, and the route must stop there.
We consider, in this case, the order segment $< +\boldsymbol{y},\ +\boldsymbol{z} >$ is the standard one.

$<\star$-operation, Stop Lemma $>$

In treating N-systems, we need diagrams of various types. And various surgeries are operated on the diagrams. We call them by a general name $\star$-*operations*. It is notable that many of $\star$-operations satisfy the following Lemma.

(Stop Lemma) If a $\star$-operation is done on some $\star$-place of a diagram, then the route which goes through the place must stop at some near vertex of the diagram.

This Lemma has a very ambiguous form. The distance between the $\star$-place and the stopping place is various. If we consider (PB) as an example of Lemma, there is exactly one arc which has $+\boldsymbol{x}$ between the two vertices.
As for Dynkin Diagrams, this Lemma has various forms. For example, notable fact is the existence of E_m $(m = 6, 7, 8)$ and the nonexistence of E_9.
And in spite of the ambigous nature of Lemma, it is very useful to explain some arithmetical phenomena. In particular,it helps in treating some "Problems of All".

M2. Trank and Star Principle

The phenomena i.e. $\star$-effects and Stop Lemma seem to work more general situations. We call it *Trunk and Star Principle* . This Principle consists of three parts:

U1. Frequently an N-system S is made of members of two types: i.e. $S(\mathrm{Tr})$: *Trunk set* and $S(\star)$: *Star Set*. And they obey somewhat different rules.

U2. In usual $S(\mathrm{Tr})$ has a clear structure, for example a *parametrizable* set. On the other hand, the structure of $S(\star)$ is usually difficult. But in many cases $|S(\star)| < \infty$, and this finite property is induced by a Stop Lemma.

U3. In many cases, in spite of U1 and U2, the structure of $S(\star)$ depends deeply on that of $S(\mathrm{Tr})$.

This principle seems to have *unexpectedly* a wide range of application. But we confine here to an example. Namely we take up P4.

Take some $(a, b, c) \in N^3$ whose GCD $= 1$. We denote $f = ax^2+by^2+cz^2$ and $W = W(f)$ of (7). Now we proceed as follows:

Step 1. Take $s, t(\geq 2) \in N$, and u with $0 \leq u \leq t-1$. Next we take two subsets L and M of $\bar{N}$. We consider the set of the following form:

$$\{\ s^{\lambda}(t\mu + u) \mid \lambda \in L,\ \mu \in M\}. \tag{8}$$

Step 2. For a given $\sigma = (s, t, u)$, we define the following three kinds of sets.

1. (λ, μ)-type: $\mathrm{Tr}(\sigma; \infty, \infty) =$ (8) with $L = M = \bar{N}$,
2. λ-type: $\mathrm{Tr}(\sigma; \infty, M) =$ (8) with $L = \bar{N}$, $|M| < \infty$,
3. μ-type: $\mathrm{Tr}(\sigma; L, \infty)=$ (8) with $M = \bar{N}$, $|L| < \infty$.

We call either of these set simply a *Trunk set with* σ , and denote it $\mathrm{Tr}(\sigma)$. We define

$$\Sigma = \{\, \sigma \,|\, W \supset \mathrm{Tr}(\sigma)\ \}.$$

For $\sigma \in \Sigma$, we define $W(\sigma)$ separating by the type of $\mathrm{Tr}(\sigma) \subset W$.

1. If $\mathrm{Tr}(\sigma)$ is of type (λ, μ), we put $W(\sigma) = \mathrm{Tr}(\sigma)$.

2. If $\mathrm{Tr}(\sigma)$ is of type λ, we define $M = \{c \in \bar{N} \mid W \supset \mathrm{Tr}(\sigma; \infty, c)\}$. And we put $W(\sigma) = \mathrm{Tr}(\sigma; \infty, M)$.
3. If $\mathrm{Tr}(\sigma)$ is of type μ, we define $L = \{c \in \bar{N} \mid W \supset \mathrm{Tr}(\sigma; c, \infty)\}$. And we put $W(\sigma) = \mathrm{Tr}(\sigma; L, \infty)$.

Finally we put $W(\mathrm{Tr})$ = the union of $W(\sigma)$'s which are maximal. Then our bold conjecture is

Conjecture A. We put $W(\star) = W - W(\mathrm{Tr})$. Then $W(\star)$ is a finite set.

Example. Take $f = x^2 + 3y^2 + 8z^2$. Then it seems plausible (no proof) that

$$W(\mathrm{Tr}) = \{16\mu + 2\} \cup \{16\mu + 6\} \cup \{16\mu + 10\} \cup \{4^\lambda(16\mu + 14)\} \quad (\lambda, \mu \in \bar{N}).$$

$$W(\star) = \{5, 23, 55, 77, 173, 187, 377, 647\}.$$

Three remarks: (1) The last set of $W(\mathrm{Tr})$ has 4^λ as a factor.
(2)Note that $\{16\mu + 2 \mid \mu \in \bar{N}\} \supset \{9^\lambda \cdot 2 \mid \lambda \in \bar{N}\}$.
(3) Max of $W(\star)$ seems to be 647. The numbers up to 1800 are searched.

Remarks: (a) Conjecture A is a bold one. But many numerical examples support it. And a similar framework works in treating P3.

(b) A trunk set of λ-type does not appear in Ramanujan's list. But this type easily slips out of computation (D1). And it is notable that, in Davenport [3], he showed that every member of $\{16^\lambda \cdot 31 | \lambda \in \bar{N}\}$ is a sum of exactly 16 fourth powers of natural numbers.

Step 3. We study the structure of $W(\star)$ on the following another assumption.

Conjecture B. The structure of $W(\star)$ depends deeply on that of $W(\mathrm{Tr})$. Thus we study the nature of $n \in W(\star)$, by expressing n by (8) with $\sigma \in \Sigma$.

This plan may seem to lack foundation. But the author has found several curious facts in a state of preliminary investigation. Probably the goal is far away. But our prospects are:

(1) A Stop Lemma of new type will appear with respect to distributions of two sets L and M.

(2) Our hope is to obtain an interaction theory of trunk sets. If W contains two $W(\sigma)$'s with different t's, $n \in W(\star)$ has different expressions of type (8).

M3. Make use of a Q-plane, Cell Principle

The member of a N-system usually tend to infinity. As noted in D2, it is difficult to see through arithmetic properties of large numbers. To overcome the difficulty, it is effective, in some cases, to let $N \to Q$.

(1) In some sequences $\{a_i\}$ $i = 1, 2 \ldots$ with $a_n \to \infty$, the ratio a_{i+1}/a_i may remain finite.

(2) By making $N \to Q$, we may apply in some cases an approximation theory, in particular the theory of continued fraction.

We show here an example of a drastic effect of making $N \to Q$.

Step 1. Consider P3. In the following, we assume the validity of N-basis Lemma for $(a, b, c, d) \in N^4$ and M. P3 has a symmetric nature with respect to the four numbers. But we take up one of them, say a. Fix three numbers B, C, D for which $1 \leq B, C, D \leq a - 1$.

We put

$$H = \{(b, c, d) \in N^3 \mid b \equiv B,\ c \equiv C,\ d \equiv D \pmod{a}\ \}$$

Step 2. Let π be a YZ-plane. We define a mapping
$\varphi : H \to \pi$ by $\varphi(b, c, d) = (b/d, c/d)$.
For $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$, we define a line on π as follows:

$$r(\boldsymbol{u}) : u_2 Y + u_3 Z + u_4 = 0.$$

Step 3. Take $(b, c, d) \in H$, and we define $C(b, c, d)$ as follows:

(case 1) If (a, b, c, d) is of type 1 of N-basis Lemma, we put
$C(b, c, d) =$ the triangle whose edges are $\{r(\boldsymbol{x}), r(\boldsymbol{y}), r(\boldsymbol{z})\}$.
(case 2) If (a, b, c, d) is of type 2, we put
$C(b, c, d) =$ the tetragon whose edges are $\{r(\boldsymbol{x}), r(\boldsymbol{y}), r(\boldsymbol{z}), r(\boldsymbol{w})\}$.
As easily seen, $(b/d,\ c/d) \in C(b, c, d)$. We call $C(b, c, d)$ a *cell.*

(Cell Principle) Assume $(\tilde{b}, \tilde{c}, \tilde{d}) \in H$ and that φ-image of this triple is in $C(b, c, d)$. Let $\tilde{M}$ for $(a, \tilde{b}, \tilde{c}, \tilde{d})$ have the same meaning of M for (a, b, c, d). Then we have $M = \tilde{M}$.

The true reason of Cell Principle is, if it exists, not yet clear. It is rather a surprise that similar phenomena appear for some other N-systems (cf. for example [21]).

M4. Semi-Local

Finally we propose a concept which we hope to be useful in treating D4. But it is a new-born one and remains yet in vague form. Our hope is to unify the following three facts by a concept "*Semi-Local*".

(A) As noted in D4, a greedy algorithm does not work for a complicated extremal problems. For example in trying to construct an odd covering (P1), we must seek a good distribution of m's on a tree T1. But it is difficult to ascertain a good *global* distribution. Thus it will be a better tactic to seek a *semi-locally* good one. Namely our hope is to fill the "?" of the following diagram: local : greedy-algorithm = semi-local : ? .

(B) As noted in M1, the location of a $\star$-place and the stopping place is important and subtle. Namely it has a nature of semi-local property.

(C) In treating some N-system, we meet the following phenomenon. About a sequence $a_1, a_2, \ldots, a_m$, we wish to determine a_i inductively on i. But it turns out that a_i is affected by a_{i+1} and a_{i+2}. And on the other hand, if we start from a *false* a_i, we cannot get to a_{i+2}. We wish to say this property as *+2 semilocal.*

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