Extensions and induced characters of some 2-groups

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1 Introduction

Let G be a finite group. We denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) the set of faithful irreducible characters of G.

Let Q_n and D_n denote the generalized quaternion group and the dihedral group, respectively, of order $2^{n+1}(n \geq 2)$. Let SD_n denote the semidihedral group of order $2^{n+1}(n \geq 3)$. As is well-known, the 2-groups $Q_n,\ D_n$ and SD_n have remarkable properties among all 2-groups (cf. [3]). Furthermore, in [4], we know that these kinds of 2-groups play an important role in character theory of 2-groups. In this note we investigate $\mathrm{Irr}(H)$ for $H = Q_n,\ D_n$ and SD_n. The main theorems are following (cf. Remark 1):

Theorem 3 *Let $n \geq 3$ and $\phi \in \mathrm{FIrr}(Q_n)$. Let G be an extension of Q_n such that $[G : Q_n] = 2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_1(Q_n)$.*

Theorem 4 *Let $n \geq 3$ and $\phi \in \mathrm{FIrr}(D_n)$. Let G be an extension of D_n such that $[G : D_n] = 2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_1(D_n)$.*

The 2-groups $G_1(Q_n)$ and $G_1(D_n)$ are defined in Section 3. These results might characterize Q_n $(n \geq 3)$ and D_n $(n \geq 3)$ among all 2-groups (see Remark 2 and Section 5).

2 Induced characters of $H = Q_n,\ D_n$ and SD_n

We use the following notation from now on.

- The generalized quaternion group $Q_n = \langle a,\ b\rangle$ $(n \geq 2)$ with
  $a^{2^n} = 1, \qquad b^2 = a^{2^{n-1}}, \qquad bab^{-1} = a^{-1}.$
- The dihedral group $D_n = \langle a,\ b\rangle$ $(n \geq 2)$ with
  $a^{2^n} = 1, \qquad b^2 = 1, \qquad bab^{-1} = a^{-1}.$
- The semidihedral group $SD_n = \langle a,\ b\rangle$ $(n \geq 3)$ with
  $a^{2^n} = 1, \qquad b^2 = 1, \qquad bab^{-1} = a^{-1+2^{n-1}}.$

We denote by ζ_{2^n} a primitive 2^nth root of unity. For $H = Q_n$ or D_n, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(1 \leq \nu < 2^{n-1})$ of H, which are not linear:

$\phi_\nu(a^i) = \zeta_{2^n}^{\nu i} + \zeta_{2^n}^{-\nu i}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leq i \leq 2^n).$

For $H = SD_n$, there are $2^{n-1} - 1$ irreducible characters ϕ_ν $(-2^{n-2} \leq \nu \leq 2^{n-2}$ for odd ν, $1 \leq \nu < 2^{n-1}$ for even $\nu)$ of H, which are not linear:

$\phi_\nu(a^i) = \zeta_{2^n}^{\nu i} + \zeta_{2^n}^{\nu i(-1+2^{n-1})}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leq i \leq 2^n).$

Each irreducible character ϕ_ν of Q_n or D_n or SD_n is induced from a linear character η_ν of the maximal normal cyclic subgroup $\langle a\rangle$: $\eta_\nu(a^i) = \zeta_{2^n}^{\nu i}$ $(1 \leq i \leq 2^n)$. For $H = Q_n$ or D_n or SD_n, an irreducible character ϕ_ν of H is faithful if and only if ν is odd. The faithful irreducible characters ϕ_ν of H are algebraically conjugate to each other, and $\mathbf{Q}(\phi_\nu) = \mathbf{Q}(\zeta_{2^n} + \zeta_{2^n}^{-1})$ when $H = Q_n$ or D_n and $\mathbf{Q}(\phi_\nu) = \mathbf{Q}(\zeta_{2^n} - \zeta_{2^n}^{-1})$ when $H = SD_n$.

We have the following.

Theorem 1 ([3, Theorem 1]) *Let $H = Q_n$ or D_n or SD_n. Let G be a 2-group which contains H with $[G : H] = 2^r$ $(r \geq 1)$. Let ϕ be a faithful irreducible character of H. Suppose that the induced character $\chi = \phi^G$ is irreducible. Then $r \leq n-2$, $\mathbf{Q}(\chi) = \mathbf{Q}(\zeta_{2^{n-r}} + \zeta_{2^{n-r}}^{-1})$, and $[\mathbf{Q}(\phi) : \mathbf{Q}(\chi)] = [G : H] = 2^r$. Moreover, the values of ϕ^G depend only on r.*

In Yamada [6], Theorem 1 is proved for the case $H = Q_n$. But the same proof is valid also for $H = D_n$ and SD_n with slight modification. We can also prove Theorem 1 mainly using Character Theory (cf. Theorem 10). We have the following which says about 2-groups G_r with $[G : H] = 2^r$, where

$H = Q_n$ or D_n or SD_n. It is easy to see that Theorem 1 follows from the following. In paticular, it follows from (1) and (2) in Proposition 2 that the values of ϕ^G depend only on r.

Proposition 2 ([3, Proposition 1]) *Let $H = Q_n$ $(n \geq 3)$ or D_n $(n \geq 3)$ or SD_n. Let G be a 2-group which contains H with $[G : H] = 2^r$ $(r \geq 1)$. Let ϕ be a faithful irreducible character of H. Suppose that the induced character $\chi = \phi^G$ is irreducible. Let*

$$H = G_0 \subset G_1 \subset \cdots \subset G_r = G, \quad [G_i : G_{i-1}] = 2,$$

$$G_i = G_{i-1} \cup u_i G_{i-1}, \quad u_i^2 \in G_{i-1} \ (1 \leq i \leq r).$$

Then $r \leq n-2$, and

(1) $\phi^{G_i}(a^{2^i z}) = 2^i \phi(a^{2^i z}), \quad z \in \mathbf{Z},\ 0 \leq i \leq r,$
(2) $\phi^{G_i}(y) = 0$ *for* $y \in (G_i - \langle a^{2^i} \rangle), \quad 0 \leq i \leq r,$
(3) $\sigma(\langle a^{2^i} \rangle) = \langle a^{2^i} \rangle$ *for any* $\sigma \in \mathrm{Aut}(G_i), \quad 0 \leq i \leq r,$
(4) $u_i a^{2^{i-1}} u_i^{-1} = a^{2^{i-1}(1+2^{n-i})}$ *for an appropriate* $u_i \in G_i, \quad 1 \leq i \leq r,$
(5) $x^{2^i} \in \langle a^{2^i} \rangle$ *for any* $x \in G_i, \quad 1 \leq i \leq r.$

Remark 1 Let $H = Q_2$ or D_2 and $\phi \in \mathrm{FIrr}(H)$. It is easy to see that there is no 2-group G such that $[G : H] = 2$ and $\phi^G \in \mathrm{Irr}(G)$.

Let $H = Q_n$ or D_n or SD_n. The following examples show that for each r $(1 \leq r \leq n-2)$, there exists $G_r \rhd H$ such that $[G_r : H] = 2^r$ and $\phi^{G_r} \in \mathrm{Irr}(G_r)$, where ϕ is a faithful irreducible character of H.

Example 1 ([3, Examples 1–2]) Let $G = \langle a,\ b,\ c \rangle$ with $a^{2^n} = 1,\ b^2 = c^{2^{n-2}} = a^{2^{n-1}},\ bab^{-1} = a^{-1},\ cac^{-1} = a^5,\ cbc^{-1} = b$. Then G has subgroups $G_r = \langle a,\ b,\ c^{2^{n-2-r}} \rangle$ $(1 \leq r \leq n-2)$. We have $[G_r : H] = 2^r$ and $H \subset G_1 \subset G_2 \subset \cdots \subset G_{n-2} = G$ for $H = Q_n = \langle a,\ b \rangle$ or $SD_n = \langle a,\ bc^{2^{n-3}} \rangle$.

Replacing $b^2 = c^{2^{n-2}} = a^{2^{n-1}}$ with $b^2 = c^{2^{n-2}} = 1$ and doing Q_n with D_n in Example 1, we have an example for $H = D_n$ or SD_n.

3 Extensions of $H = Q_n,\ D_n$ and SD_n

In this section we determine the extensions G of H such that $[G:H]=2$ or 4 and $\phi^G \in \mathrm{Irr}(G)$, where $H = Q_n$ or D_n or SD_n and $\phi \in \mathrm{FIrr}(H)$.

Let $G_1(Q_n)$ and $G_1(D_n)$ be the groups with generators and relations as follows:

- $G_1(Q_n) = \langle a,\ b,\ u_1 \rangle$ with
  $a^{2^n} = 1, b^2 = a^{2^{n-1}}, bab^{-1} = a^{-1}, u_1au_1^{-1} = a^{-1+2^{n-1}}, u_1b = bu_1, u_1^2 = 1.$
- $G_1(D_n) = \langle a,\ b,\ u_1 \rangle$ with
  $a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1},\ u_1au_1^{-1} = a^{-1+2^{n-1}},\ u_1b = bu_1,\ u_1^2 = 1.$

It is easy to see that $G_1(Q_n)$ is not isomorphic to $G_1(D_n)$ by comparing the number of involutions of $G_1(Q_n)$ and $G_1(D_n)$. We have

$$G_1(Q_n) \supset Q_n = \langle a,\ b \rangle, \quad SD_n = \langle a,\ u_1 \rangle,$$

$$G_1(D_n) \supset D_n = \langle a,\ b \rangle, \quad SD_n = \langle a,\ u_1 \rangle.$$

Then, if $\phi \in \mathrm{FIrr}(Q_n)$ or $\mathrm{FIrr}(SD_n)$, then $\phi^{G_1(Q_n)} \in \mathrm{Irr}(G_1(Q_n))$ and if $\phi \in \mathrm{FIrr}(D_n)$ or $\mathrm{FIrr}(SD_n)$, then $\phi^{G_1(D_n)} \in \mathrm{Irr}(G_1(D_n))$.

We have the following.

Theorem 3 ([3, Theorem 3]) *Let $n \geq 3$ and $\phi \in \mathrm{FIrr}(Q_n)$. Let G be an extension of Q_n such that $[G:Q_n]=2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_1(Q_n)$.*

Theorem 4 ([3, Theorem 3]) *Let $n \geq 3$ and $\phi \in \mathrm{FIrr}(D_n)$. Let G be an extension of D_n such that $[G:D_n]=2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_1(D_n)$.*

Theorem 5 ([3, Theorem 4]) *Let $\phi \in \mathrm{FIrr}(SD_n)$. Let G be an extension of SD_n such that $[G:SD_n]=2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_1(Q_n)$ or $G_1(D_n)$.*

Next for $n \geq 4$, we define four kinds of 2-groups:

- $G_2^{(1)}(Q_n) = \langle a,\ b,\ u_1,\ u_2 \rangle$ ($\triangleright\, G_1(Q_n) = \langle a,\ b,\ u_1 \rangle$) with
  $a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1},\ u_1au_1^{-1} = a^{-1+2^{n-1}},\ u_1b = bu_1,$
  $u_1^2 = 1,\ u_2au_2^{-1} = a^{1+2^{n-2}},\ u_2b = bu_2,\ u_2u_1 = u_1u_2,\ u_2^2 = bu_1.$

- $G_2^{(2)}(Q_n) = \langle a,\ b,\ u_1,\ u_2\rangle$ ($\triangleright\, G_1(Q_n) = \langle a,\ b,\ u_1\rangle$) with
$a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1},\ u_1au_1^{-1} = a^{-1+2^{n-1}}, u_1b = bu_1,$
$u_1^2 = 1,\ u_2au_2^{-1} = a^{1+2^{n-2}}bu_1,\ u_2b = bu_2,\ u_2u_1 = u_1u_2,\ u_2^2 = 1.$

- $G_2^{(1)}(D_n) = \langle a,\ b,\ u_1,\ u_2\rangle$ ($\triangleright\, G_1(D_n) = \langle a,\ b,\ u_1\rangle$) with
$a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1},\ u_1au_1^{-1} = a^{-1+2^{n-1}},\ u_1b = bu_1,\ u_1^2 = 1,$
$u_2au_2^{-1} = a^{1+2^{n-2}},\ u_2b = bu_2,\ u_2u_1 = u_1u_2,\ u_2^2 = bu_1.$

- $G_2^{(2)}(D_n) = \langle a,\ b,\ u_1,\ u_2\rangle$ ($\triangleright\, G_1(D_n) = \langle a,\ b,\ u_1\rangle$) with
$a^{2^n} = b^2 = 1,\ bab^{-1} = a^{-1},\ u_1au_1^{-1} = a^{-1+2^{n-1}},\ u_1b = bu_1,\ u_1^2 = 1,$
$u_2au_2^{-1} = abu_1,\ u_2b = u_1u_2,\ u_2u_1 = bu_2,\ u_2^2 = 1.$

It is easy to see that these four kinds of 2-groups are not isomorphic to each other by comparing the number of involutions of these groups. Then we have the following.

Theorem 6 ([3, Theorem 5]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(Q_n)$. Let G be an extension of Q_n such that $[G : Q_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(Q_n)$ or $G_2^{(2)}(Q_n)$.*

Theorem 7 ([3, Theorem 5]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(D_n)$. Let G be an extension of D_n such that $[G : D_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(D_n)$ or $G_2^{(2)}(D_n)$.*

Theorem 8 ([3, Theorem 6]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(SD_n)$. Let G be an extension of SD_n such that $[G : SD_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(Q_n)$ or $G_2^{(2)}(Q_n)$ or $G_2^{(1)}(D_n)$ or $G_2^{(2)}(D_n)$.*

Remark 2 The author dose not know an example of a pair of a 2-group H and $\phi \in \mathrm{FIrr}(H)$ having exactly one extension G such that $[G : H] = 2$ and $\phi^G \in \mathrm{Irr}(G)$, except for Q_n $(n \geq 3)$ and D_n $(n \geq 3)$ (cf. Theorems 3 and 4).

4 Induced characters of finite groups

Let H be a finite group and $\phi \in \mathrm{Irr}(H)$. Let F be a set of finite groups G such that $G \supset H$ and $\phi^G \in \mathrm{Irr}(G)$. Put

$$m(\phi, H, \mathrm{F}) = \max\{[G : H] \mid G \in \mathrm{F}\}.$$

We have a problem: how large $m(\phi, H, \mathrm{F})$ is? For example, we have

$$m(\phi, H, \mathrm{F}_2) = 2^{n-2}$$

for the case $H = Q_n$ or D_n or SD_n, $\phi \in \mathrm{FIrr}(H)$, and F_2 is the set of all the 2-groups G such that $G \supset H$ and $\phi^G \in \mathrm{Irr}(G)$ (cf. Theorem 1).

If F is the set of finite groups G such that $G \triangleright H$ and $\phi^G \in \mathrm{Irr}(G)$, then we have the following. We denote the automorphism group of H and the inner automorphism group of H with $\mathrm{Aut}(H)$ and $\mathrm{Inn}(H)$, respectively.

Theorem 9 ([3, Theorem 2]) *Let G be a finite group and H a normal subgroup of G. Let $\phi \in \mathrm{Irr}(H)$ and suppose that $\phi^G \in \mathrm{Irr}(G)$. Then*

$$[G : H] \leq [\mathrm{Aut}(H) : \mathrm{Inn}(H)].$$

There is an example of 2-groups H and G such that $G \triangleright H$, $\phi \in \mathrm{Irr}(H)$, $\phi^G \in \mathrm{Irr}(G)$ and $[G : H] = [\mathrm{Aut}(H) : \mathrm{Inn}(H)]$. (See [3, Example 3].)

Again, let $H = Q_n$ $(n \geq 3)$ or D_n or SD_n, and let $\phi \in \mathrm{FIrr}(H)$. It is easy to see that $[\mathrm{Aut}(H) : \mathrm{Inn}(H)] = 2^{n-1}$ (respectively, 2^{n-2}) for $H = Q_n$ or D_n (respectively, for $H = SD_n$). Combining this with Theorem 1, we have $m(\phi, H, \mathrm{F}_2) \leq [\mathrm{Aut}(H) : \mathrm{Inn}(H)]$, where F_2 is the set of all the 2-groups G such that $G \supset H$ and $\phi^G \in \mathrm{Irr}(G)$. This inequality gives rise to following question: Is it true that $H \triangleleft G$ for any $G \in \mathrm{F}_2$?

It follows from Theorems 6–8 that this is false. In fact, Q_n is not normal in $G_2^{(2)}(Q_n)$, D_n is not normal in $G_2^{(2)}(D_n)$ and SD_n is not normal in $G_2^{(2)}(Q_n)$ and $G_2^{(2)}(D_n)$. In paticular, these examples of $n = 4$ also show there exists a 2-group G such that $m(\phi, H, \mathrm{F}_2) = [G : H]$ and H is not normal in G.

In general, we have the following (cf. [5, Corollary (2.30)]).

Theorem 10 *Let G be a finite group and H a subgroup of G. Let $\phi \in \mathrm{Irr}(H)$ and suppose that $\phi^G \in \mathrm{Irr}(G)$. Then*

$$[G : H] \leq \frac{|H|}{\phi(1)^2}.$$

5 Appendix

The 2-groups $D_n,\ Q_n$ and SD_n have a normal cyclic subgroup of index 2. But we have another 2-group M_n of order 2^{n+1} with this property:

$$M_n = \langle a,\ b \mid a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{1+2^{n-1}} \rangle, \quad (n \geq 3),$$

which may be called an *ordinary* 2-group. These 2-groups have also faithful irreducible characters. And those characters are algebraccally conjugate to each other. We determined, in [2], all the 2-groups G such that $G \supset M_n$, $[G : M_n] = 2$ and $\phi^G \in \mathrm{Irr}(G)$ for any $\phi \in \mathrm{FIrr}(M_n)$. That is: there are eight such groups G for $n \geq 4$ and there four such groups G for $n = 3$. We remark $G^{(8-1)}$ and $G^{(8-2)}$ in Theorem 6 in [2] don't satisfy the property that $\phi^G \in \mathrm{Irr}(G)$. This result is in a contrast to results for $Q_n,\ D_n$ and SD_n.

Theorem 11 ([2, Theorem 5]) *For $n \geq 4$, let $M_n = \langle a,\ b \rangle$ with $a^{2^n} = 1,\ b^2 = 1$ and $bab^{-1} = a^{1+2^{n-1}}$. Let G be an extension of M_n such that $[G : M_n] = 2$ and that $\phi^G \in \mathrm{Irr}(G)$, where ϕ is a faithful irreducible character of M_n. Then G is isomorphic to one of the following groups:*

$G^{(1)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1},\ ubu^{-1} = b,\ u^2 = 1.$
$G^{(2)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1},\ ubu^{-1} = b,\ u^2 = a^{2^{n-1}}.$
$G^{(3)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{1+2^{n-2}},\ ubu^{-1} = b,\ u^2 = b.$
$G^{(4)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1+2^{n-2}},\ ubu^{-1} = b,\ u^2 = b.$
$G^{(5-1)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = ab,\ ubu^{-1} = b,\ u^2 = 1.$
$G^{(6)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1+2^{n-2}}b,\ ubu^{-1} = b,\ u^2 = 1.$
$G^{(7)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1+2^{n-2}}b,\ ubu^{-1} = b,\ u^2 = a^{2^{n-1}}.$
$G^{(8)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1}b,\ ubu^{-1} = b,\ u^2 = b.$

Theorem 12 ([2, Theorem 6]) *For $n = 3$, let $M_n = \langle a,\ b \rangle$ with $a^{2^n} = 1,\ b^2 = 1$ and $bab^{-1} = a^{1+2^{n-1}}$. Let G be an extension of M_n such that $[G : M_n] = 2$ and that $\phi^G \in \mathrm{Irr}(G)$, where ϕ is a faithful irreducible character of M_n. Then G is isomorphic to one of the following groups:*

$G^{(1)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1},\ ubu^{-1} = b,\ u^2 = 1.$
$G^{(2)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = a^{-1},\ ubu^{-1} = b,\ u^2 = a^{2^{n-1}}.$
$G^{(5-1)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = ab,\ ubu^{-1} = b,\ u^2 = 1.$
$G^{(5-2)} = \langle a,\ b,\ u \rangle$ *with* $uau^{-1} = ab,\ ubu^{-1} = b,\ u^2 = a^{2^{n-1}}.$

By the free software “GAP”we determined, for all 2-group H of order 2^r $(r \leq 6)$ having a faithful irreducible character and $\phi \in \mathrm{FIrr}(H)$, the number of the 2-groups G such that $G \supset H$, $[G:H]=2$ and $\phi^G \in \mathrm{Irr}(G)$. And we showed that, for all 2-group H of order 2^7 having a faithful irreducible character and $\phi \in \mathrm{FIrr}(H)$, the number of the 2-groups G such that $G \supset H$, $[G:H]=2$ and $\phi^G \in \mathrm{Irr}(G)$ isn't one. We note there exist 2-groups H having ϕ and θ $(\in \mathrm{FIrr}(H))$ which are not algebrally conjugate to each other.

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