New 5-Designs with Automorphism Group $PSL(2,23)$

北詰 正顕 (千葉大学)

宗政 昭弘 (九州大学)

1997 年 10 月 23 日

当時岡山大学の大学院生だった，原田昌章さんから $5-(24,10,36)$ デザインを見つけたという話を聞いたのは 1996 年の 10 月頃でした。詳しく聞いてみると，このデザインのブロックは，Mathieu 群 M_{24} を自己同型群に持つ Steiner system $S(5,8,24)$ のブロック (octads と呼ばれる) に 2 点を付け加えた形になっており，かつ，デザイン全体に $PSL(2,23)$ が作用しているということを知りました。しかし，原田さんたちの証明は，完全に計算機によるもので，何故これがデザインになるのか，ということについて答えるものではありませんでした。そのとき思ったことは，octad に付け加える 2 点についてうまい記述を与えれば，これから 5-design ができることの証明は computer を用いずともできるのではないか，ということでした。ところが，これが難しい。それどころか，理詰めで考えようとすると，5-design であることが嘘のようにすら思えてくるのでした。

1ヶ月後の山形大の集会で，やはりこの話題に興味を持った九州大学の宗政さんと情報交換を行って，まず，付け加える 2 点の取り方が，$PSL(2,23)$ の作用を除いて一意的であることを computer を用いて確かめることから始めて，類似の design が作れないかどうかなどを調べる作業を試みるようになりました。その結果，ひとつの $PSL(2,23)$ -orbit でデザインになるのは，原田さんたちのもの以外にないこと，類似という意味で，やはり $S(5,8,24)$ に関連する概念である dodecade (12 点集合で $5-(24,12,48)$ デザインをつくる) から 2 点を除くというようなことからはデザインは出てこないこと，などがわかりました。それでは，$PSL(2,23)$-orbits をいくつか合わせてはどうか，というのが宗政さんの着想で，そういうものを求める作業が始まったのです。

ひとつ見つかった $(5-(24,10,72)$ デザイン) というメールを宗政さんから受け取ったのは，11 月 21 日の朝でした。私の方でも計算はしていたのですが，今となっては理由も分からない寄り道 (無為な作業) をしていて何の発見もなかったときでした。不思議なもので (あるいは，よくあるように) あるとわかれば，一直線にいくらでも見つかるものです。その日の午後には，第一次の結果として，$PSL(2,23)$-orbits の和集合で 5-design になるもの (の候補) 28 個が出そろいすぐにそれが確かめられました。

そこで，次の目的として $PGL(2,23)$-orbits を考えることにしたのです。$PGL(2,23)$-orbits は全部で 37 個で，これらをどう組み合わせるかという問題で，単純に考えれば 2^{37} 通りの場合分けになります。結果的にこの数字は，我々の力量から見て適当なものだったと思われます。つまり，直ちに計算できるほど少なくはなく，絶望的になるほどは大きくないという数字でした。千葉大の岡田靖則さんにプログラムの援助をお願いし，はじめのバージョンでは 2 ヶ月位かかるかという見通しだったこともありますが，最終的には 72 時間程度で全ての場合をチェックすることができました。結果として，485,742 個の 5-designs が得られたのです。これらの自己同型群は $PSL(2,23)$ か $PGL(2,23)$ になり，

当初からわかっているものより大きくなることはありません。証明には，二重可移群の分類という高級な定理を使っています。回避することもできるかもしれませんが，こだわらないことにしました。自己同型群が決まるので，そこから，得られたデザインが互いに非同型であることも得られます。

振り返って見れば，これらのことには $S(5,8,24)$ の対称性が大きく影響を与えているのだと改めて感じます。既に触れたように，dodecade から 2 点を除く，というようなところからは何一つ出てきませんでした。また，膨大な数のデザインが出てきましたが，その中で唯一，single orbit からなる原田さん達のデザインが一層際だって目立つように思えます。何故これが 5-design になるのかという問いには，結局答えられなかったわけです。当初「computer を使わなくても，$S(5,8,24)$ の性質を使えば ...」というところから始まった話が，最後は computer に強く依存する話になってしまったのは皮肉なことと思います。

より詳しい内容については，この後に続く投稿中の preprint にゆだねることにします。ここまでの文章については (講演内容と同様に) 北詰に責任があることを最後に付記いたします。

New 5-Designs with Automorphism Group $PSL(2, 23)$

Masaaki Kitazume
Department of Mathematics and Informatics
Chiba University
and
Akihiro Munemasa
Graduate School of Mathematics
Kyushu University

July 22, 1997

1 Introduction

A t-(v, k, λ) design is a set of v points together with a collection of its k-subsets called blocks, so that any t points are contained in exactly λ blocks. The Steiner system $S(5, 8, 24)$ which is a unique 5-$(24, 8, 1)$ design, has automorphism group the Mathieu group M_{24}. Blocks of this Steiner system are called octads. The group $PSL(2, 23)$ can be embedded into M_{24} in such a way that it acts transitively on the set of octads. The stabilizer of an octad B in $PSL(2, 23)$ has 20 orbits on the set of 2-subsets of the complement of B. It turns out that only one of the orbits on 2-subsets has the following property: if we take a representative $\{x, y\}$, then the $PSL(2, 23)$-orbit containing $B \cup \{x, y\}$ forms a 5-$(24, 10, 36)$ design. This design was first discovered by Gulliver and Harada [3] as the support of the codewords of weight 10 in the lifted Golay code over $\mathbf{Z}_4$. The purpose of this paper is to enumerate all 4- and 5-designs obtained by a union of $PSL(2, 23)$-orbits on 10-subsets containing an octad. To be more precise, let B be an octad. The group $PSL(2, 23)$ has 20 orbits on the set of 10-subsets containing B. One of the 20 orbits (this is not the one which forms a 5-design) is invariant under $PGL(2, 23)$, the normalizer of $PSL(2, 23)$ in the symmetric group S_{24}. The remaining 19 orbits have their 'duals', i.e., if $\mathcal{O}$ is one of the remaining 19 orbits and $C \in \mathcal{O}$, then the $PGL(2, 23)$-orbit containing C is the disjoint union $\mathcal{O} \cup \mathcal{O}^\tau$, where τ is an element of $PGL(2, 23)$ not in $PSL(2, 23)$. Therefore we obtain 39 $PSL(2, 23)$-orbits of 10-subsets. We then compute the collapsed incidence matrices of t-subsets and the 39 $PSL(2, 23)$-orbits for $t = 4, 5$, collapsed under the action of $PSL(2, 23)$. With these incidence matrices, it is easy by a computer search to enumerate all 4- and 5-designs obtained by a union of some of these 39 $PSL(2, 23)$-orbits. In particular, we have found 4-designs and 5-designs with the following parameters.

Theorem 1. *There exist* 5-$(24, 10, \lambda)$ *designs admitting* $PSL(2, 23)$ *as automorphism group for* $\lambda = 36, 72$ *and* $18m$*, where* m *is a positive integer with* $6 \leq m \leq 52$*.*

Theorem 2. *There exist* 4-$(24, 10, \lambda)$ *designs admitting* $PSL(2, 23)$ *as automorphism group for* $\lambda = 60m$*, where* m *is a positive integer with* $2 \leq m \leq 55$*.*

Apart from the 5-$(24, 10, 36)$ design found by Gulliver and Harada [3], very few designs with these parameters were previously known to exist. Up to taking the complement of the set of blocks of a t-(v, k, λ) design, it suffices to consider the case where the number of blocks is at most half the number of all k-subsets in a v-element set. This means that we may restrict our attention to the case $\lambda \leq \frac{1}{2}\binom{20}{6}$ when we consider 4-$(24, 10, \lambda)$ designs. Similarly for 5-$(24, 10, \lambda)$ designs, we restrict our attention to the case $\lambda \leq \frac{1}{2}\binom{19}{5}$. According to [2], 4-$(24, 10, \lambda)$ designs are known to exist for $\lambda = 600$, 1800 and 3360, while 5-$(24, 10, \lambda)$ designs are known to exist for $\lambda = 180$, 540 and 1008 only. Note that 5-$(24, 10, \lambda)$ designs with $\lambda = 540, 1008$ are obtained as orbits of M_{24} acting on the set of 10-element subsets (see [4]), so by taking the union of these two designs we obtain a 5-$(24, 10, 1548)$ design which is also a 4-$(24, 10, 5160)$ design.

There are, in general, a large number of designs with parameters given in Theorem 2 and Theorem 1. Using the fact that $PGL(2, 23)$ is the normalizer of $PSL(2, 23)$ in the symmetric group S_{24}, we can show that the designs we enumerated are not isomorphic unless they are $PGL(2, 23)$-conjugate. This will enable us to count the number of designs up to isomorphism. The results are tabulated in section 3.

2 Construction of $PSL(2, 23)$-orbits

The group $G = PSL(2, 23)$ is the permutation group on the projective line $\mathcal{P} = GF(23) \cup \{\infty\}$, generated by the following two permutations: $x \mapsto x + 1$, $x \mapsto -\frac{1}{x}$, where we understand $\infty + 1 = \infty$, $\pm\frac{1}{\infty} = 0$, $\pm\frac{1}{0} = \infty$. Note that G is 3-homogeneous, that is, G acts transitively on the set of 3-subsets of $\mathcal{P}$. Let

$$B = \{2, 7, 8, 11, 13, 14, 18, 20\}.$$

Then the G-orbit containing B has length 759, forming a 5-$(24, 8, 1)$ design. The setwise stabilizer G_B of B in G has 20 orbits on the set of 10-subsets of $\mathcal{P}$ containing B. Equivalently, G_B has 20 orbits on the set of 2-subsets of $\mathcal{P} \setminus B$. Representatives and the lengths for each of the 20 orbits are given in Table 1.

Number	0	1	2	3	4	5	6	7	8	9
$\{x, y\}$	10,∞	1,∞	12,∞	5,∞	15,∞	16,∞	3,∞	21,∞	9,∞	5,10
Length	6072	6072	6072	6072	6072	6072	6072	6072	6072	6072
Number	10	11	12	13	14	15	16	17	18	19
$\{x, y\}$	4,∞	23,∞	19,∞	17,∞	10,12	10,15	3,10	10,21	10,16	6,∞
Length	3036	3036	3036	3036	3036	3036	3036	3036	3036	3036

Table 1: G_B-orbits of 2-subsets of $\mathcal{P} \setminus B$

Let $\mathcal{O}_i$ be the G-orbit containing $B_i = B \cup \{x_i, y_i\}$, where x_i, y_i are the elements given in the column i of the row '$\{x, y\}$' of Table 1. The entries in the row 'Length' are the lengths of the orbits $\mathcal{O}_i$.

The group $PGL(2, 23)$ is the permutation group on $\mathcal{P}$ generated by G together with the involution $\tau : x \mapsto -x$, where we understand $-\infty = \infty$. The group $PGL(2, 23)$ contains G as a subgroup of index 2. The orbit $\mathcal{O}_{19}$ is invariant under τ, hence invariant under $PGL(2, 23)$. All other orbits $\mathcal{O}_i$ $(0 \le i \le 18)$ are not invariant under $PGL(2, 23)$, so the $PGL(2, 23)$-orbit containing B_i is $\mathcal{O}_i \cup \mathcal{O}_i^\tau$. Note that $\mathcal{O}_i^\tau$ is the G-orbit containing B_i^τ. To construct the collapsed incidence matrices, we first decompose the set of t-subsets of $\mathcal{P}$ into G-orbits for $t = 4$ and 5. Representatives are given in Table 2, Table 3, respectively.

Orbit	Representative
$T_1^{(4)}$	$\{1, 2, 3, 4\}$
$T_2^{(4)}$	$\{1, 2, 3, 5\}$
$T_3^{(4)}$	$\{1, 2, 3, 7\}$
$T_4^{(4)}$	$\{1, 2, 3, 9\}$
$T_5^{(4)}$	$\{1, 2, 3, 10\}$

Table 2

Orbit	Representative
$T_1^{(5)}$	$\{1, 2, 3, 4, 5\}$
$T_2^{(5)}$	$\{1, 2, 3, 4, 7\}$
$T_3^{(5)}$	$\{1, 2, 3, 4, 8\}$
$T_4^{(5)}$	$\{1, 2, 3, 4, 9\}$
$T_5^{(5)}$	$\{1, 2, 3, 4, 11\}$
$T_6^{(5)}$	$\{1, 2, 3, 4, 14\}$
$T_7^{(5)}$	$\{1, 2, 3, 5, 8\}$

Table 3

Let Ω denote the set of 39 G-orbits $\mathcal{O}_i$ $(0 \le i \le 19)$ and $\mathcal{O}_i^\tau$ $(0 \le i \le 18)$. We now give the collapsed incidence matrix of 4-subsets and Ω. The rows are indexed by the orbits $T_1^{(4)}, \ldots, T_5^{(4)}$ of 4-subsets, the columns are indexed by Ω. Its entry at position $(T, \mathcal{O})$ is the number of elements of $\mathcal{O}$ containing a representative for T. This number is independent of the choice of a representative for T. The entries are tabulated in Table 4.

The definition of the collapsed incidence matrix of 5-subsets and Ω is analogous, and its entries are tabulated in Table 5.

It follows immediately from Table 5 that $\mathcal{O}_0$ and $\mathcal{O}_0^\tau$ form (isomorphic) 5-$(24, 10, 36)$ designs.

For a subset Θ of Ω, we denote the set of blocks $\cup_{\mathcal{O} \in \Theta} \mathcal{O}$ by $\mathcal{B}(\Theta)$. Note that $\mathcal{B}(\Theta)$ forms a 4-design if and only if the submatrix of Table 4 consisting of the rows corresponding to Θ has a constant row sum. Similarly $\mathcal{B}(\Theta)$ forms a 5-design if and only if the submatrix of Table 5 consisting of the rows corresponding to Θ has a constant row sum. It is now a simple computer search to enumerate all 4- and 5-designs obtained in this way.

We exhibit examples of Θ for some values of λ in Table 6 for $t = 5$, in Table 7 for $t = 4$. These examples are sufficient to prove Theorem 1 and Theorem 2. In Table 6 and Table 7, $\mathcal{O}_i$ is abbreviated as i, $\mathcal{O}_i^\tau$ is abbreviated as i', and $\{\mathcal{O}_i, \mathcal{O}_i^\tau\}$ is abbreviated as i^* to save space.

Proof of Theorem 1. Table 6 lists examples of Θ for some values of λ, whose validity can be checked directly by Table 5. Observe that none of Θ given in Table 6 contains $\mathcal{O}_0$

	$\mathcal{O}_0$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	$\mathcal{O}_6$	$\mathcal{O}_7$	$\mathcal{O}_8$	$\mathcal{O}_9$	$\mathcal{O}_{10}$	$\mathcal{O}_{11}$	$\mathcal{O}_{12}$
$T_1^{(4)}$	120	130	122	122	122	118	118	116	116	112	64	65	63
$T_2^{(4)}$	120	114	122	120	120	122	122	122	118	122	62	59	60
$T_3^{(4)}$	120	124	112	120	120	124	120	128	128	116	56	62	54
$T_4^{(4)}$	120	108	124	116	116	116	116	120	120	128	52	54	54
$T_5^{(4)}$	120	120	116	120	120	120	124	116	124	128	60	56	66

	$\mathcal{O}_{13}$	$\mathcal{O}_{14}$	$\mathcal{O}_{15}$	$\mathcal{O}_{16}$	$\mathcal{O}_{17}$	$\mathcal{O}_{18}$	$\mathcal{O}_{19}$	$\mathcal{O}_0^\tau$	$\mathcal{O}_1^\tau$	$\mathcal{O}_2^\tau$	$\mathcal{O}_3^\tau$	$\mathcal{O}_4^\tau$	$\mathcal{O}_5^\tau$
$T_1^{(4)}$	60	59	58	60	57	55	63	120	130	122	122	122	118
$T_2^{(4)}$	63	63	57	56	60	59	59	120	114	122	120	120	122
$T_3^{(4)}$	60	58	60	60	62	58	58	120	108	124	116	116	116
$T_4^{(4)}$	56	66	68	68	66	74	58	120	124	112	120	120	124
$T_5^{(4)}$	58	52	62	60	58	60	60	120	120	116	120	120	120

	$\mathcal{O}_6^\tau$	$\mathcal{O}_7^\tau$	$\mathcal{O}_8^\tau$	$\mathcal{O}_9^\tau$	$\mathcal{O}_{10}^\tau$	$\mathcal{O}_{11}^\tau$	$\mathcal{O}_{12}^\tau$	$\mathcal{O}_{13}^\tau$	$\mathcal{O}_{14}^\tau$	$\mathcal{O}_{15}^\tau$	$\mathcal{O}_{16}^\tau$	$\mathcal{O}_{17}^\tau$	$\mathcal{O}_{18}^\tau$
$T_1^{(4)}$	118	116	116	112	64	65	63	60	59	58	60	57	55
$T_2^{(4)}$	122	122	118	122	62	59	60	63	63	57	56	60	59
$T_3^{(4)}$	116	120	120	128	52	54	54	56	66	68	68	66	74
$T_4^{(4)}$	120	128	128	116	56	62	54	60	58	60	60	62	58
$T_5^{(4)}$	124	116	124	128	60	56	66	58	52	62	60	58	60

Table 4: collapsed incidence matrix for $t = 4$

	$\mathcal{O}_0$	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	$\mathcal{O}_4$	$\mathcal{O}_5$	$\mathcal{O}_6$	$\mathcal{O}_7$	$\mathcal{O}_8$	$\mathcal{O}_9$	$\mathcal{O}_{10}$	$\mathcal{O}_{11}$	$\mathcal{O}_{12}$
$T_1^{(5)}$	36	43	40	38	38	35	35	33	31	28	25	23	21
$T_2^{(5)}$	36	40	33	35	37	34	35	34	38	37	17	18	17
$T_3^{(5)}$	36	41	31	37	36	39	37	40	42	32	18	20	17
$T_4^{(5)}$	36	38	37	37	36	36	34	37	33	32	15	20	17
$T_5^{(5)}$	36	29	40	34	33	33	34	34	36	41	15	14	17
$T_6^{(5)}$	36	35	35	37	37	37	40	33	35	41	20	16	24
$T_7^{(5)}$	36	26	36	34	35	38	37	41	37	41	16	15	13

	$\mathcal{O}_{13}$	$\mathcal{O}_{14}$	$\mathcal{O}_{15}$	$\mathcal{O}_{16}$	$\mathcal{O}_{17}$	$\mathcal{O}_{18}$	$\mathcal{O}_{19}$	$\mathcal{O}_0^\tau$	$\mathcal{O}_1^\tau$	$\mathcal{O}_2^\tau$	$\mathcal{O}_3^\tau$	$\mathcal{O}_4^\tau$	$\mathcal{O}_5^\tau$
$T_1^{(5)}$	20	19	14	14	14	12	21	36	43	40	38	38	35
$T_2^{(5)}$	14	18	21	22	18	18	18	36	40	33	35	37	34
$T_3^{(5)}$	19	13	17	15	17	15	18	36	29	40	34	33	33
$T_4^{(5)}$	19	19	17	21	20	18	18	36	38	37	37	36	36
$T_5^{(5)}$	16	19	23	21	20	27	18	36	41	31	37	36	39
$T_6^{(5)}$	19	14	16	16	16	15	18	36	35	35	37	37	37
$T_7^{(5)}$	19	24	18	17	21	21	15	36	26	36	34	35	38

	$\mathcal{O}_6^\tau$	$\mathcal{O}_7^\tau$	$\mathcal{O}_8^\tau$	$\mathcal{O}_9^\tau$	$\mathcal{O}_{10}^\tau$	$\mathcal{O}_{11}^\tau$	$\mathcal{O}_{12}^\tau$	$\mathcal{O}_{13}^\tau$	$\mathcal{O}_{14}^\tau$	$\mathcal{O}_{15}^\tau$	$\mathcal{O}_{16}^\tau$	$\mathcal{O}_{17}^\tau$	$\mathcal{O}_{18}^\tau$
$T_1^{(5)}$	35	33	31	28	25	23	21	20	19	14	14	14	12
$T_2^{(5)}$	35	34	38	37	17	18	17	14	18	21	22	18	18
$T_3^{(5)}$	34	34	36	41	15	14	17	16	19	23	21	20	27
$T_4^{(5)}$	34	37	33	32	15	20	17	19	19	17	21	20	18
$T_5^{(5)}$	37	40	42	32	18	20	17	19	13	17	15	17	15
$T_6^{(5)}$	40	33	35	41	20	16	24	19	14	16	16	16	15
$T_7^{(5)}$	37	41	37	41	16	15	13	19	24	18	17	21	21

Table 5: collapsed incidence matrix for $t = 5$

λ	Θ
108	$4, 6, 11, 18$
126	$3, 12, 13, 14, 15^*$
216	$3^*, 4^*, 7, 8'$
234	$2^*, 5, 6, 8', 10', 16^*$
324	$1, 3, 4^*, 5', 6', 7^*, 8'$
342	$1, 2, 3^*, 4, 7', 8', 9', 10', 17^*$
432	$1^*, 3^*, 5^*, 6, 7^*, 8', 9', 14, 19$
450	$1^*, 2^*, 4^*, 7^*, 8, 9^*, 13', 17, 19$
540	$1^*, 3^*, 5^*, 6, 7^*, 8, 9^*, 11^*, 13, 14, 15, 19$
558	$1, 2^*, 3', 4^*, 5, 6^*, 7^*, 8^*, 9, 11^*, 16$
648	$1^*, 2^*, 3^*, 4^*, 5, 6^*, 7, 8^*, 9, 13', 14, 15', 17^*, 19$
666	$1^*, 2^*, 3^*, 5^*, 6^*, 7^*, 8^*, 9^*, 11^*, 14, 16', 19$
756	$1^*, 2^*, 3^*, 4^*, 5^*, 6^*, 7, 8^*, 9, 10, 13^*, 14, 15^*, 16', 17^*, 19$
774	$1^*, 2^*, 3, 4^*, 5^*, 6^*, 7, 8^*, 9^*, 10', 11^*, 13^*, 14, 16^*, 18^*, 19$
846	$1^*, 2^*, 3^*, 5^*, 6^*, 7^*, 8, 9^*, 10^*, 11^*, 12^*, 13', 14^*, 15^*, 16^*, 17, 18^*, 19$
864	$1^*, 2^*, 3, 4^*, 5^*, 6^*, 7^*, 8, 9^*, 10^*, 11^*, 12, 13^*, 14, 15^*, 16^*, 17, 18^*, 19$

Table 6: examples of 5-$(24, 10, \lambda)$ designs

λ	Θ
180	$12, 14, 15'$
3000	$1^*, 2^*, 3^*, 4^*, 5, 6^*, 7^*, 8^*, 9^*, 10, 11^*, 12^*, 13^*, 14^*, 15, 16^*, 17, 18^*, 19$
3060	$1^*, 2^*, 3^*, 4^*, 5^*, 6^*, 7^*, 8^*, 9^*, 10, 11^*, 12^*, 13, 14^*, 15', 16^*, 17^*, 18, 19$

Table 7: examples of 4-$(24, 10, \lambda)$ designs

or $\mathcal{O}_0^\tau$, and that each of $\mathcal{O}_0$ and $\mathcal{O}_0^\tau$ becomes a 5-$(24, 10, 36)$ design. Thus we may add $\mathcal{O}_0$ or $\{\mathcal{O}_0, \mathcal{O}_0^\tau\}$ to the set Θ to construct a 5-$(24, 10, \lambda + 36)$ design and a 5-$(24, 10, \lambda + 72)$ design. Designs with λ not given in Table 6 but given in Theorem 1 can be found in this way, and we therefore obtain designs with all parameters given in Theorem 1.

Proof of Theorem 2. Note that a 5-$(24, 10, \lambda)$ design is also a 4-$(24, 10, 10\lambda/3)$ design. Theorem 1 implies the existence of 4-$(24, 10, \lambda)$ designs with $\lambda = 120, 240$, $6m$ with $6 \leq m \leq 52$. It remains to show the existence of such designs with $\lambda = 180, 300$, $6m$ with $53 \leq m \leq 55$. Table 7 lists examples of Θ for $\lambda = 180$, $6m$ with $m = 50, 51$, none of them containing $\mathcal{O}_0$ or $\mathcal{O}_0^\tau$. Since each of $\mathcal{O}_0$ and $\mathcal{O}_0^\tau$ becomes a 4-$(24, 10, 120)$ design, we may add $\mathcal{O}_0$ or $\{\mathcal{O}_0, \mathcal{O}_0^\tau\}$ to the set Θ to construct a 4-$(24, 10, \lambda + 120)$ design and a 4-$(24, 10, \lambda + 240)$ design. Thus, 4-$(24, 10, \lambda + 120)$ design with $\lambda = 300$, $6m$ with $53 \leq m \leq 55$ can be constructed from examples given in Table 7. We therefore obtain designs with all parameters given in Theorem 2.

3 Isomorphisms among the designs

Indeed, we have enumerated all 4-designs and 5-designs obtained as a union of orbits in Ω. There are a large number of designs with the same set of parameters, and one might ask when these designs are isomorphic. It turns out that this question can be answered as soon as we determine the automorphism groups of these designs.

Theorem 3. *Let Θ be a subset of Ω. Suppose Θ is different from $\{\mathcal{O}_i | 0 \leq i \leq 19\}$ or $\{\mathcal{O}_i^\tau | 0 \leq i \leq 19\}$. Then the automorphism group of the design $(\mathcal{P}, \mathcal{B}(\Theta))$ is $PGL(2, 23)$ if Θ is invariant under τ, $PSL(2, 23)$ otherwise.*

Proof. Since $G = PSL(2,23)$ leaves each $\mathcal{O}_i$ and $\mathcal{O}_i^\tau$ invariant, the automorphism group contains G. We claim that, other than the alternating group A_{24}, there exist exactly three proper subgroups of the symmetric group S_{24} which properly contain G: $PGL(2,23)$ and two embeddings of M_{24}. This can be seen as follows. By the classification of doubly-transitive groups [1], such a subgroup is isomorphic to $PGL(2,23)$ or M_{24}. Let us count the number of pairs $(H, \tilde{H})$, where H and $\tilde{H}$ are subgroups of S_{24} satisfying $H \subset \tilde{H}$, $H \cong PSL(2,23)$, $\tilde{H} \cong M_{24}$. Note that M_{24} is unique up to conjugacy and self-normalizing in S_{24}. Note also that $PSL(2,23)$ is unique up to conjugacy and self-normalizing in M_{24}. Thus the number of such pair is $|S_{24} : M_{24}||M_{24} : PSL(2,23)| = |S_{24} : PSL(2,23)|$. On the other hand, $PSL(2,23)$ is unique up to conjugacy in S_{24} and its normalizer is $PGL(2,23)$. Thus the number of choices for H is $|S_{24} : PGL(2,23)| = \frac{1}{2}|S_{24} : PSL(2,23)|$. It follows that, given H, there are exactly two choices for $\tilde{H}$, proving our claim. The two embeddings M_{24}, which are obtained as $\mathrm{Aut}(\mathcal{P}, B^G)$ and $\mathrm{Aut}(\mathcal{P}, (B^G)^\tau)$, act transitively on $\mathcal{B}(\{\mathcal{O}_i | 0 \le i \le 19\})$, $\mathcal{B}(\{\mathcal{O}_i^\tau | 0 \le i \le 19\})$, respectively. Thus, if $\mathcal{B}(\Theta)$ is neither of these, then the automorphism group of $(\mathcal{P}, \mathcal{B}(\Theta))$ is either G or $PGL(2,23)$. Clearly, the automorphism group is $PGL(2,23)$ if and only if Θ is invariant under τ. □

Corollary 4. *Let Θ, Θ' be subsets of Ω. Suppose Θ, Θ' are different from $\{\mathcal{O}_i | 0 \le i \le 19\}$ or $\{\mathcal{O}_i^\tau | 0 \le i \le 19\}$. Then the designs $\mathcal{D} = (\mathcal{P}, \mathcal{B}(\Theta))$ and $\mathcal{D}' = (\mathcal{P}, \mathcal{B}(\Theta'))$ are isomorphic if and only if $\Theta' \in \{\Theta, \Theta^\tau\}$.*

Proof. By Theorem 3, the automorphism groups $\mathrm{Aut}\,\mathcal{D}$ and $\mathrm{Aut}\,\mathcal{D}'$ are either $PSL(2,23)$ or $PGL(2,23)$. Suppose that there is an isomorphism f from $\mathcal{D}$ to $\mathcal{D}'$. Then $\mathrm{Aut}\,\mathcal{D}$ and $\mathrm{Aut}\,\mathcal{D}'$ are not only isomorphic, but identical groups. It follows that f normalizes the characteristic subgroup $PSL(2,23)$ of $\mathrm{Aut}\,\mathcal{D}$, so $f \in PGL(2,23)$. This implies $\Theta' \in \{\Theta, \Theta^\tau\}$. □

Corollary 4 gives a simple criterion for two designs of the form $(\mathcal{P}, \mathcal{B}(\Theta))$ to be isomorphic. Thus, from a computer enumeration of subsets Θ of Ω for which $(\mathcal{P}, \mathcal{B}(\Theta))$ is a 4- or 5-design, we can easily count the number of 4- or 5-designs up to isomorphism. Table 8, Table 9 give the number of 4-designs, 5-designs, respectively, with full automorphism group $PSL(2,23)$ or $PGL(2,23)$ of the form $(\mathcal{P}, \mathcal{B}(\Theta))$ for some $\Theta \subset \Omega$ up to isomorphism.

We conclude the paper by noting that no 6-design has been found among the 5-designs of the form $(\mathcal{P}, \mathcal{B}(\Theta))$, $\Theta \subset \Omega$.

λ	$PGL(2,23)$	$PSL(2,23)$
120	0	1
180	0	2
240	1	2
300	0	6
360	1	23
420	0	37
480	2	115
540	0	218
600	2	522
660	0	1119
720	9	2335
780	0	4195
840	4	7755
900	0	13555
960	15	23431
1020	0	37340
1080	12	58810
1140	0	85441
1200	18	123173
1260	0	167114
1320	23	226483
1380	0	285792
1440	36	361750
1500	0	421394
1560	27	492497
1620	0	536281
1680	41	589258
1740	0	598295
1800	28	619299
1860	0	579164
1920	34	553012
1980	0	483250
2040	31	429241
2100	0	353089
2160	30	296076
2220	0	224605
2280	21	174897
2340	0	122424
2400	18	88142
2460	0	57606
2520	7	38895
2580	0	23151
2640	9	14413
2700	0	7743
2760	3	4296
2820	0	2120
2880	2	1172
2940	0	473
3000	1	254
3060	0	72
3120	0	43
3180	0	9
3240	0	4
3300	0	1

Table 8

λ	$PGL(2,23)$	$PSL(2,23)$
36	0	1
54	0	0
72	1	1
90	0	0
108	0	7
126	0	2
144	1	20
162	0	28
180	0	45
198	0	100
216	3	185
234	0	248
252	0	689
270	0	691
288	4	1783
306	0	2038
324	2	3716
342	0	4929
360	3	8211
378	0	9082
396	2	15526
414	0	14714
432	4	23725
450	0	21430
468	0	30825
486	0	26361
504	6	42919
522	0	27825
540	0	52256
558	0	27210
576	6	41169
594	0	23963
612	3	26131
630	0	16889
648	3	19357
666	0	10178
684	4	12572
702	0	6096
720	1	5975
738	0	3105
756	1	2341
774	0	1163
792	1	1052
810	0	356
828	0	416
846	0	139
864	0	117
882	0	45
900	0	14
918	0	4
936	0	4
954	0	0
972	0	0
990	0	0

Table 9

Acknowledgments

The authors would like to thank Masaaki Harada for helpful discussions, and Yasunori Okada for his help in computer programming.

References

[1] P. J. Cameron, Finite pemutation groups and finite simple groups, Bull. London Math. Soc., 13 (1981), 1–22.

[2] C. J. Colbourn and J. H. Dinitz, The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, 1996.

[3] T. A. Gulliver and M. Harada, Extremal double circulant Type II codes over $\mathbf{Z}_4$ and construction of 5-$(24, 10, 36)$ designs, submitted.

[4] E. S. Kramer, S. S. Magliveras and D. M. Mesner, t-Designs from the large Mathieu groups, Discrete Math., 36 (1981), 171–189.